Isogeometric Topology Optimization by Using Optimality Criteria and Implicit Function

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ABSTRACT

A new method for structural topology optimization is introduced which employs the Isogeometric Analysis (IA) method. In this approach, an implicit function is constructed over the whole domain by Non-Uniform Rational B-Spline (NURBS) basis functions which are also used for creating the geometry and the surface of solution of the elasticity problem. Inspiration of the level set method; zero level of the function describes the boundary of the structure. An optimality criterion is derived to improve the implicit function towards the optimum boundaries. The last section of this paper is devoted to some numerical examples in order to demonstrate the performance of the method as well as the concluding remarks.

1. INTRODUCTION

During the last two decades topology optimization methods have been improved by many researchers. In pioneering procedures on structural topology optimization, introduced by Bendsøe and Kikuchi [1], the results are an array of material density in the finite elements which are used for discretization of the domain of interest. In such element based methods the material density function is constant within each finite element. The jogged boundaries and checkerboarding patterns are the well known drawbacks of such methods especially when a coarse mesh and lower order finite elements are used [2]. Several remedies are introduced to prevent these instabilities and keep the results applicable [2-3].

As the next generation of topology optimization methods, nodal based methods were

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developed that have manifested the capability of removing the mentioned instabilities in element based methods. In these approaches the material densities are determined at the discretization nodes or points by considering a function throughout the domain. Level set methods, by Sethian, Wang and Allaire [4-6], and topology optimization by implicit function and regularization, by Belytschko [7], use this technique for topology optimization. On the other hand, it has been shown that these methods can be used when the mesh-less methods are employed for analyzing the structure [8].

Recently, the Isogeometric Analysis (IA) method has been utilized for topology optimization of structures instead of the FE approach where the NURBS basis functions have also been used for approximating the material density function by Hassani *et al* [9-10]. Density function shows the distribution of material throughout the domain of interest and is approximated by NURBS basis functions. Control points are restricted to be within the zero (for empty areas or voids) and one (for solid areas) interval. Apart from preventing checkerboard patterns and mesh dependency, the IA method has shown to be well-matched with topology optimization problems. Since the density function varies between zero and one the porous media might appear in optimum layout. Although, it is alleviated by the power law in SIMP method [11, 12], it is not totally removed.

In this article, following [9] in order to prevent the porous media, inspired from level set methods [4-6] as well as topology optimization by implicit function [7], a nodal based method is presented. In this approach, the recently developed IA approach is employed for analysis and an implicit function is constructed over the whole domain by NURBS basis functions where zero level of the function describes the boundary of the structure. An updating scheme is suggested to improve the implicit (density) function towards finding the optimum boundaries. Although an optimality criteria method is used in this paper, different optimization methods such as Methods of Moving Asymptotes (MMA) [13] and heuristic methods [14-16] can be utilized for this purpose.

IA is a relatively new method proposed and developed by Hughes and his co-workers in recent years [17-21]. This method is a logical extension and generalization of the classical finite element method and has many features in common with it. However, it is more geometrically based and takes inspiration from Computer Aided Geometry Design (CAGD). A primary goal of IA is to be geometrically precise no matter how coarse the discretization beside simplification of mesh refinement by eliminating the need for communication with the CAD geometry once the initial model is constructed. The main idea of the method is to use the same basis functions which are employed for geometry description for approximation and interpolation of the unknown field variables as well. Due to some interesting properties of B-splines and NURBS, they are perfect candidates for this purpose.

The outline of this paper is as follows. In Section 2, the IA method for plane stress problems is briefly explained. Section 3 is devoted to the concise definition of the topology optimization problem as well as creating the implicit function and derivation of the optimality criteria. In Section 4 a few examples are presented to demonstrate the performance of the method. Finally, the results are discussed in the last section.

2. ISOGEOMETRICAL ANALYSIS

By recent developments in the CAGD technology, the geometrical definition and generation of complex surfaces and objects have become achievable [22]. For this purpose, Splines and some modified versions of them, i.e. NURBS and T-Splines, are commonly employed. The main idea in the IA method is that any component of a field variable which satisfies a governing partial differential equation, i.e. the solution, is imagined as a (hyper-) surface that can be constructed by the proper versions of splines [23,24]. For example, in displacement method for elasticity problems, each of the components of the displacement vector is considered as a surface which can be constructed by NURBS and the defining parameters of these surfaces are sought. The criteria for finding these parameters can be obtained by minimizing a total potential energy functional or equivalently by implementation of the virtual work principle.

The IA method has some features in common with other numerical methods such as finite elements and mesh-free methods. Discretization of the domain of interest is performed by using the control points of splines instead of, for instance, using the finite element meshes, finite difference grids, or collection of points in the meshfree methods. Also, the basis functions of these splines are used both for approximation of the unknown variables as well as for interpolation.

The procedure of the IA for elasticity problems is comprised of the following steps. First, the geometry of the domain of interest is constructed by using the NURBS technology. Depending on the complexity of the geometry and topology of the problem, multiple NURBS patches can be used in this stage. These patches may be thought of as kind of macro elements in the finite element method and can be assembled in the same fashion [17]. In the next step, borrowing the ideas from isoparametric finite elements, the geometry as well as the displacement components are approximated by making use of the NURBS basis functions. Then, following a standard procedure like the weighted residuals or the variational methods, or similarly using the principle of virtual work, the approximated quantities are substituted into the obtained relations. This will result in a system of linear equations to be solved. One should note that following this procedure the control variables are evaluated and to obtain the displacements at certain points a kind of post processing is required. A brief introduction to the construction of NURBS surfaces followed by derivation of IA formulation for plane elasticity problems are the subjects of the next two subsections.

2.1 Surface and volume definition by NURBS

A NURBS surface is parametrically constructed as [22]

$$S(r,s) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \frac{N_{i,p_1}(r) N_{j,p_2}(s) \, \omega_{i,j}}{\sum_{k=0}^{n_1} \sum_{l=0}^{n_2} N_{k,p_1}(r) N_{l,p_2}(s) \, \omega_{k,l}} P_{i,j} \tag{1}$$

Where $P_{i,j}$ are $(n_1+1) \times (n_2+1)$ control points, $\omega_{i,j}$ are the associated weights, and $N_{i,p_1}(r)$ and $N_{j,p_2}(s)$ are the normalized B-splines basis functions of degree p_1 and p_2,

respectively. The i-th B-spline basis function of degree p_1, denoted by $N_{i,p_1}(r)$, is defined recursively as:

$$N_{i,0}(r) = \begin{cases} 1 & if \quad r_i \leq r \leq r_{i+1} \\ 0 & otherwise \end{cases} \quad N_{i,p_1} = \frac{r-r_i}{r_{i+p_1}-r_i}N_{i,p_1-1}(r) + \frac{r_{i+p_1+1}-r}{r_{i+p_1+1}-r_{i+1}}N_{i+1,p_1-1}(r) \quad (2)$$

Where $\mathbf{r} = \{r_0, r_1, ..., r_{m_1}\}$ is the knot vector and r_i are a non-decreasing sequence of real numbers, which are called knots. The knot vector $\mathbf{s} = \{s_0, s_1, ..., s_{m_2}\}$ is employed to define the $N_{j,p_2}(s)$ basic functions for other direction. The interval $[r_0, r_{m_1}] \times [s_0, s_{m_2}]$ forms a Patch [17]. A knot vector, for instance in r direction, is called open if the first and last knots have a multiplicity of p_1+1. In this case, the number of knots is equal to $m_1 = n_1 + p_1 + 1$. Also, the interval $[r_i, r_{i+1})$ is called a knot span where at most p_1+1 of the basic functions $N_{i,p_1}(r)$ are non-zero which are $N_{i-p_1,p_1}(r)$, ..., $N_{i,p_1}(r)$. For more details Reference [17] can be consulted.

2.2 Numerical formulation for plane elasticity problems

In the IA method, the domain of problem might be divided into subdomains or patches so that B-spline or NURBS parametric space is local to these patches. A patch is like an element in the finite element method and the approximation of unknown function can be written over a patch. Therefore, the global coefficient matrix, which is similar to the stiffness matrix in elasticity problems, can be constructed by employing the conventional assembling which is used in the finite element method.

By using the NURBS basis functions for a patch p, the approximated displacement functions $\mathbf{u}^p = [u, v]$ can be written as

$$\mathbf{u}^p(r,s) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} R_{i,j}(r,s)\mathbf{u}_{i,j}^p \quad (3)$$

Where $R_{i,j}(r,s)$ is the rational term in Equation (1). It should be noted that the geometry is also approximated by B-spline basis functions as

$$\mathbf{x}^p(r,s) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} R_{i,j}(r,s)\mathbf{x}_{i,j}^p \quad (4)$$

By using the local support property of NURBS basis functions, the above relation can be summarized as it follows in any given $(r,s) \in [r_i, r_{i+1}) \times [s_j, s_{j+1})$.

$$\mathbf{u}^p(r,s) = \sum_{e=i-p_1}^{i} \sum_{f=j-p_2}^{j} R_{e,f}(r,s)\mathbf{u}_{e,f}^p = \mathbf{RU} \qquad (5)$$

$$\mathbf{x}^p(r,s) = \sum_{e=i-p_1}^{i} \sum_{f=j-p_2}^{j} R_{e,f}(r,s)\mathbf{x}_{e,f}^p = \mathbf{RX} \qquad (6)$$

The strain-displacement matrix $\mathbf{B}$ can be constructed from the following fundamental equations

$$\boldsymbol{\varepsilon} = \mathbf{Du} \;\; \rightarrow \;\; \boldsymbol{\varepsilon} = \mathbf{BU} \qquad (7)$$

Where $\mathbf{D}$ is the differential operation matrix. Following a standard approach for the derivation of the finite elements formulation, the matrix of coefficient can easily be obtained. For example, by implementing the virtual displacement method with existence of body forces $\mathbf{b}$ and traction forces $\mathbf{t}$ we can write

$$\int_{V^p} \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma} dV - \int_{V^p} \delta\mathbf{u}^T \mathbf{b} dV - \int_{\Gamma^p} \delta\mathbf{u}^T \mathbf{t} d\Gamma = 0, \qquad (8)$$

Where V^p and Γ^p are the volume and the boundary of patch p.

Now, by substituting $\delta\boldsymbol{\varepsilon} = \mathbf{B}\delta\mathbf{U}$ from Equation (7) and the constitutive equation $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$, in Equation (8) and by dropping the non-zero coefficient of $\delta\mathbf{U}^T$, the matrix of coefficients can be obtained as

$$\mathbf{K}^p = \int_{V^p} \mathbf{B}^T \mathbf{CB} dV \qquad (9)$$

3. TOPOLOGY OPTIMIZATION PROBLEM

The problem at hand is defined as finding the stiffest possible structure when a certain amount of material is given. A structure with maximum global stiffness provides a minimum for the external work with the real displacement field or minimum mean compliance. Since, minimization of mean compliance is equivalent to the maximization of the total potential energy, the topology optimization problem can be constructed as below [3, 25].

$$
\begin{aligned}
&\text{max} &&\text{min } \Pi(\mathbf{u}) \\
&\textit{subject to} && \Omega_s \le \bar{\Omega}_s \\
&\textit{and} && \textit{design restrictions}
\end{aligned}
\qquad (10)
$$

Where $\mathbf{u}$ is displacement field, Π is total potential energy and $\bar{\Omega}_s$ is the amount of

material available. Ω_s is the volume of solid material in each design. One should note that minimization of $\Pi(\mathbf{u})$ in (11) is equivalent to satisfying the state equations or equilibrium. $\Pi(\mathbf{u})$ can be written as follows

$$\Pi(\mathbf{u}) = \frac{1}{2}\int_\Omega \boldsymbol{\varepsilon}^T(\mathbf{u})\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u})d\Omega - \int_\Omega \mathbf{u}^T\mathbf{b}d\Omega - \int_\Gamma \mathbf{u}^T\mathbf{t}d\Gamma \tag{11}$$

In structural topology optimization, the problem is how to distribute the material in order to minimize the objective function. In other words, the goal can be thought of as determination of the optimal spatial material distribution. To achieve this, an implicit function can be considered over the whole domain which is able to take positive (solid) and negative (void) values during the optimization process and the zero level of the function describes the boundary of the structure. The implicit or density function for patch p can be approximated by NURBS basis functions as follows

$$\phi^p(\mathbf{r}) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} R_{i,j}(\mathbf{r})\Phi^p_{i,j} \tag{12}$$

Where $\Phi^p_{i,j}$ are the third components of control points of the NURBS surface in patch p and can be assumed as design variables of the optimization problem. It is noted that here the same basis functions are used for approximation of geometry, displacements and the density function. The boundaries, solid and void areas of the domain are described by implicit function as follows

$$\begin{aligned}\varphi(\mathbf{x}) &= 0 \quad on \quad \Gamma \\ \varphi(\mathbf{x}) &> 0 \quad inside \quad \Omega \\ \varphi(\mathbf{x}) &< 0 \quad outside \quad \Omega\end{aligned} \tag{13}$$

In order to prevent drastic changes during the optimization process $\varphi(\mathbf{x})$ is substituted by signed distance functions in the level set method or can be cut by a parameter α. By doing this, a narrow band about the surface $\varphi(\mathbf{x}) = 0$ is considered to be altered. Therefore, we impose the constraint

$$if \quad \left|\varphi(\mathbf{x})\right| > \alpha \quad then \quad \varphi(\mathbf{x}) = \alpha\, sign(\varphi(\mathbf{x})) \tag{14}$$

In order to find out the density and elasticity matrix for each point of the structure the heaviside function $H(\varphi)$ is constructed based on zero level of the implicit function $\varphi(\mathbf{x})$ as follows

$$C(\mathbf{x}) = H(\varphi(\mathbf{x}))C^0, \qquad H(\varphi) = \begin{cases} 0 & \varphi \le 0 \\ 1 & \varphi > 0 \end{cases} \tag{15}$$

where C^0 is the elasticity matrix of solid material. Since, first derivatives of the objective function and the constraints are needed in optimization algorithm and the derivation of the heaviside step function is the Dirac delta function, it makes the method unworkable [7]. In fact, there is no guarantee of having a Gauss point exactly on boundary of the structure when the Dirac delta function is integrated numerically. Therefore, the step function should be regularized so that integration of its derivative becomes numerically stable. In this research, different regularized functions for satisfying equilibrium equation and the volume constraint have been considered as follows:

$$\bar{H}_1(\varphi) = \begin{cases} 1 & \varphi > +\beta \\ (\dfrac{1}{2\beta}\varphi + \dfrac{1}{2})^{\mu} & |\varphi| \le +\beta \\ 0 & \varphi < -\beta \end{cases} \tag{16}$$

And

$$\bar{H}_2(\varphi) = \begin{cases} 1 & \varphi > +\beta \\ \dfrac{1}{2\beta}\varphi + \dfrac{1}{2} & |\varphi| \le +\beta \\ 0 & \varphi < -\beta \end{cases} \tag{17}$$

where β is assumed to be less than α. The regularization procedure is demonstrated in Figure (1). First, $\varphi(\mathbf{x})$ is truncated by α and then the border of structure which is exactly line $\varphi(\mathbf{x}) = 0$ can be extended to area Γ by choosing parameter β and defining $\bar{H}_1(\varphi)$. It is noted that since $\bar{H}_2(\varphi)$ is used for calculating the volume constraint, it does not need to be penalized by μ.
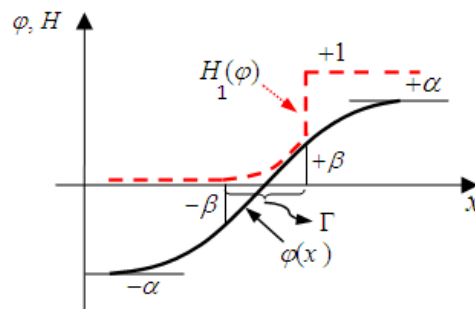

Figure 1. Regularized Heaviside function

By considering $\Phi_{i,j}$ as design variables, the optimization problem, Equation (10), can be discretized as below:

$$
\begin{aligned}
\max_{\varphi_{i,j}} \quad & \min_{i=1,...,n_1, \ j=1,...,n_2} \Pi(\mathbf{u}) \\
subject \ to \quad & \varphi_{i,j} - \alpha \le 0 \quad i=1,...,n_1, \ j=1,...,n_2 \\
& -\varphi_{i,j} - \alpha \le 0 \quad i=1,...,n_1, \ j=1,...,n_2 \\
and \quad & \Omega_s \le \bar{\Omega}_s
\end{aligned}
\tag{18}
$$

The Lagrangian function ℓ of the optimization problem can be constructed by using the Lagrange multipliers:

$$
\ell = \Pi(\mathbf{u}) - \Lambda(\Omega_s - \bar{\Omega}_s) - \sum_{i,j=1}^{n \times m = N} \lambda_1(\varphi_{i,j} - \alpha) - \\
\sum_{i,j=1}^{n \times m = N} \lambda_2(-\varphi_{i,j} - \alpha)
\tag{19}
$$

where Λ, λ_1 and λ_2 are the volume, upper and lower bound Lagrange multipliers, respectively, which are positive according to the Kuhn-Tucker conditions [25]. The stationary condition with respect to the design variables $\Phi_{i,j}$ can be obtained as follows

$$
\frac{\partial \Pi(\mathbf{u})}{\partial \varphi_{i,j}} - \Lambda \frac{\partial \Omega_s}{\partial \varphi_{i,j}} - \lambda_1 + \lambda_2 = 0
\tag{20}
$$

By manipulating the above equation it can be written as

$$
E_{i,j} = 1 + \lambda_1 / \Lambda \frac{\partial \Omega_s}{\partial \varphi_{i,j}} - \lambda_2 / \Lambda \frac{\partial \Omega_s}{\partial \varphi_{i,j}}
\tag{21}
$$

Where

$$
E_{i,j} = \frac{\frac{1}{2} \int_\Omega \frac{\partial \bar{H}_1(\varphi)}{\partial \Phi_{i,j}} \boldsymbol{\varepsilon}^T \mathbf{C} \boldsymbol{\varepsilon} \, d\Omega}{\Lambda \int_\Omega \frac{\partial \bar{H}_2(\varphi)}{\partial \Phi_{i,j}} \, d\Omega}
\tag{22}
$$

It can be assumed that in iteration k, the design variable $\Phi_{i,j}$ has been decreased in order to move towards optimum point. Therefore, $\Phi_{i,j} < \alpha$ and the upper side limit is not

active, which yields $\lambda_1 = 0$. Since $\Lambda \dfrac{\partial \Omega_s}{\partial \varphi_{i,j}}$ is a positive real number and $\lambda_2 \geq 0$, from (21) it follows that $E_{i,j}^k \leq 1$. On the other hand, increasing $\Phi_{i,j}$, results $E_{i,j}^k \geq 1$. Therefore, inspired by this argument, $E_{i,j}^k$ is calculated and compared with unity. If $E_{i,j}^k < 1$ then $\Phi_{i,j}$ is decreased by the move limit ζ and vice versa. Based on this conclusion and considering the side limits, the following resizing scheme is suggested.

$$\Phi_{i,j}^{k+1} = \Phi_{i,j}^k + \zeta(1 - \frac{1}{(E_{i,j}^k)^\eta}) \tag{23}$$

Where superscript k denotes the iteration number, ζ is move limit coefficient and η is damping factor of the resizing scheme. One should note that derivation of regularized Heaviside function in Eq. (22) can be obtained by following formula:

$$\frac{\partial \bar{H}(\varphi)}{\partial \Phi_{i,j}} = \frac{\partial \bar{H}(\varphi)}{\partial \varphi} \frac{\partial \varphi}{\partial \Phi_{i,j}} \tag{24}$$

Also in Eq. (22), $E_{i,j}^k$ depends on the current value of Λ_k which needs to be adjusted in an inner iteration loop. For this purpose the bisection method can be employed [25-26].

4. NUMERICAL EXAMPLES

To demonstrate the performance of the method two examples of isotropic plane elasticity problems are presented in this section. The modulus of elasticity and the Poisson's ratio are considered as 1500 kgf/cm^2 and 0.3, respectively. In the following examples dimensions and point load magnitude are considered as $L = 8\,cm$, $H = 5\,cm$ and $P = 100\,Kgf$, respectively.

Example 1: The geometry, loading and boundary conditions are illustrated in Figure 2. In this example the ability of the proposed method in capturing the optimum topology and the effect of the number of control points is studied. For this purpose, a couple of control nets with 400 and 1617 points are used for discretizing the design domain. The degree of the NURBS' basis functions is assumed $p = 3$. Also, optimization parameters are considered to be $\alpha = 1.7$, $\beta = 0.25$, the exponent $\mu = 3$, the damping factor $\eta = 0.5$ and the move limit coefficient $\zeta = 0.5\alpha$.

The resulted layouts are depicted in Figures 3(a) and 3(b). From the results, it is observed that obtained topology has not changed by using different numbers of control points. The results are compared to optimum layouts with the same problem definition in reference [9]. As it is shown in Figures 4(a) and 4(b) porous media emerges by using SIMP method which is not the case here. For the sake of comparison with finite element based methods, the

optimal design from Sigmund's 99 line MATLAB code [27] is illustrated in Figure 5. It should be noted that in this code noise cleaning technique is implemented in order to prevent checker-boarding and mesh dependency instabilities [3]. It can be seen that the obtained topologies in both methods are identical, however, the system of equations that shall be solved here for analysis of the structure in each iteration is much smaller.
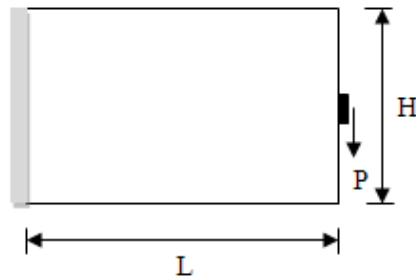

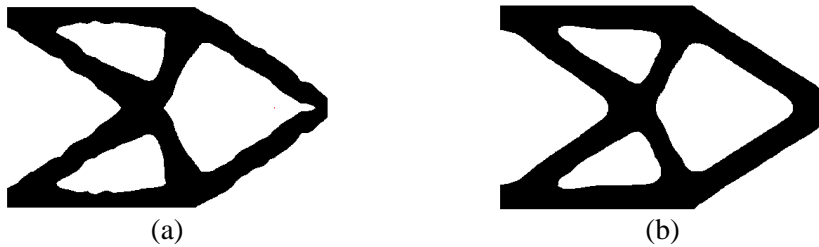Figure 2. Problem definition of example 1

(a)   (b)

Figure 3. Optimum layout (a) by using 400, (b) by using 1617 control points

(a)   (b)

Figure 4. Optimum layout (a) by using 400, (b) 1617 control points [9]

Figure 5. Optimum layout by SIMP and FEM [3,27]

Example 2: Topology optimization of the MBB beam is here considered as the second example. The design domain and boundary conditions are shown in Figure 6. In order to discretize the design domain 95 patches with 1617 control points are employed. The degree

of the NURBS' basis functions is considered to be 3. The volume fraction is taken as 40%. Equally spaced knot vectors are defined as $\mathbf{r} = \{0, 0, 0, 0, 0.5, 1, 1, 1, 1\}$, $\mathbf{s} = \{0, 0, 0, 0, 0.5, 1, 1, 1, 1\}$. Also, optimization parameters are considered to be $\alpha = 1.0$, $\beta = 0.25$, the exponent $\mu = 5$, the damping factor $\eta = 0.5$ and the move limit coefficient $\zeta = 0.5\alpha$.

The obtained layout by employing the proposed method is illustrated in Figure 7(a). For the sake of comparison, the result generated by using the SIMP material model together with the IA and the FE methods, as reported in references [9] and [3], are also shown in Figures 7(b) and 7(c), respectively. As it is observed the obtained topologies are basically identical. However, it is noticed that the gray areas do not exist in the result by the proposed method.
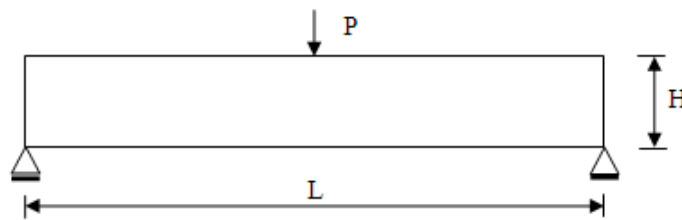

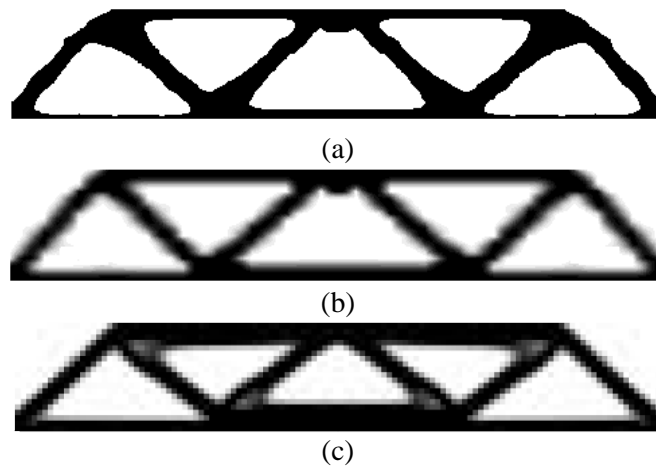Figure 6. Problem definition of example 2

(a)

(b)

(c)

Figure 7. Optimum layouts by using (a) proposed method (b) SIMP and IA [9] (c) SIMP and FEM [3]

5. CONCLUSIONS

In this paper, an implicit function is considered to form the topology of structure so that the zero level of the function indicates the boundaries of the structure. The implicit function is created by the NURBS basis functions and organized by a set of control points which the third coordinates are used as the optimization design variables. By using the IA method, geometry and displacement field are also approximated by the same basis functions. In this

work an optimality criterion is suggested to modify the implicit function during the optimization process so that the topology changes towards optimum. According to our experience, even with a coarse net of control points, one ends up with a reasonable topology that is always checkerboard-free and mesh independent which is not the case when the FE method is used. Also, it is noted that in topology optimization by the FE method, apart from requiring an appropriate discretization to have accuracy in analysis results, usually a finer mesh is needed in order to have smooth boundaries in final optimum topology. But by using the proposed method, if the desired accuracy of the analysis results is fulfilled by a number of control points they would be enough to obtain precise boundaries due to flexibility of the NURBS in constructing the implicit function.

REFERENCES

1. Bendsøe MP, Kikuchi N. Generating optimal topologies in structural design using homogenization method, *Comput Meth Appl Mech Eng* 1988; **71**: 197-224.
2. Sigmund O. Numerical instabilities in topology optimization: A survey on procedure dealing with checkerboards, mesh-dependencies and local minima, *Struct Optim* 1998; **16**: 68-75.
3. Bendsøe MP, Sigmund O. *Topology Optimization, Theory, Methods And Applications*, Springer, Germany, 2003.
4. Sethian J, Wiegmann A. Structural boundary design via level set and immersed interface methods, *J Comput Phys* 2000; **163**: 489–528.
5. Wang MY, Wang X, Guo D. A level-set method for structural topology optimization, *Comp Meth Appl Mech Eng* 2003; **192**: 227–46.
6. Allaire G, Jouve F, Toader AM. Structural optimization using sensitivity analysis and a level-set method, *J Comp Phys* 2004; **194**: 363-93.
7. Belytschko T, Xiao SP, Parimi C. Topology optimization with implicit functions and regularization, *Int J Numer Meth Eng* 2003; **57**: 1177–96.
8. Zhou JX, Zou W. Meshless approximation combined with implicit topology description for optimization of continua, *Struct Multidiscip Optim* 2008; **36**: 347–53.
9. Hassani B, Khanzadi M, Tavakkoli SM. An Isogeometrical approach to structural topology optimization by optimality criteria, *Struct Multidiscip Optim* 2012; **45**(2): 223–33.
10. Tavakkoli SM, Hassani B, Ghasemnejad H. Isogeometric Topology optimization of strucutres by using MMA, *Int J Optim Civil Eng* 2013; **3**(2): 313-26.
11. Rozvany GIN, Zhou M. The COC algorithm, Part I: Cross section optimization or sizing, *Comp Meth Appl Mech Eng* 1991; **89**: 281-308.
12. Rozvany GIN. *Structural Design via Optimality Criteria*, Kluwer Academic Publishers, Dordrecht, 1989.
13. Svanberg K. The method of moving asymptotes – a new method for structural optimization, *Int J Numer Meth Eng* 1987; **24**: 359–73.
14. Kaveh A, Hassani B, Shojaee S, Tavakkoli SM. Structural topology optimization using ant colony methodology. *Eng Struct* 2008; **30**(9): 2559–65.

15. Tavakkoli SM, Shahryari L, Parsa M. Topology optimization of space strucutres using ant colony method, *Int J Optim Civil Eng* 2013; **3**(3): 359-70.
16. Kaveh A, Kalatjari VR, Talebpour MH, Torkamanzadeh J. Configuration optimization of trusses using a multi heuristic based search method, *Int J Optim Civil Eng* 2013; **3**(1): 151-78.
17. Hughes TJR, Cottrell JA, Bazilevs Y. Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement, *Comput Meth Appl Mech Eng* 2005; **194**: 4135–95.
18. Bazilevs Y, Beirao Da Veiga L, Cottrell J, Hughes TJR, Sangalli G. Isogeometric analysis: approximation, stability and error estimates for h-refined meshes. *Math Mod Meth Appl Sci* 2006; **16**: 1031–90.
19. Bazilevs Y, Calo V, Cottrell J, Hughes TJR, Reali A, Scovazzi G. Variational multiscale residual-based turbulence modeling for large eddy simulation of incompressible flows, *Comput Meth Appl Mech Eng* 2007; **197**: 173–201.
20. Bazilevs Y, Calo VM, Zhang Y, Hughes TJR. Isogeometric fluid structure interaction analysis with applications to arterial blood flow, *Comput Meth Appl Mech Eng* 2006; **38**: 310-22
21. Cottrell JA, Reali A, Bazilevs Y, Hughes TJR. Isogeometric analysis of structural vibrations, *Comput Meth Appl Mech Eng* 2006; **195**: 5257–96.
22. Piegl L, Tiller W. *The NURBS book,* Springer-Verlag, New York, USA, 1997.
23. Hassani B, Moghaddam NZ, Tavakkoli SM. Isogeometrical solution of Laplace equation, *Asian J Civil Eng* 2009; **10**(5): 572-92.
24. Hassani B, Khanzadi M, Tavakkoli SM, Moghaddam NZ. Isogeometric shape optimization of three-dimensional problems, *8th World Congress on Structural and Multidisciplinary Optimization*, June 1-5, Lisbon, Portugal, 2009.
25. Hassani B, Hinton E. *Homogenization and Structural Topology Optimization: Theory, Practice and Software,* Springer, London, UK, 1999.
26. Suzuki K, Kikuchi N. A homogenization method for shape and topology optimization, *Comput Meth Appl Mech Eng* 1991; **93**: 291-318.
27. Sigmund O. A 99 line topology optimization code written in MATLAB, *Struct Multidiscip Optim* 2001; **21**: 120-7.