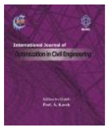
INTERNATIONAL JOURNAL OF OPTIMIZATION IN CIVIL ENGINEERING
Int. J. Optim. Civil Eng., 2016; 6(2):187-209



OPTIMIZATION OF A PRODUCTION LOT SIZING PROBLEM WITH QUANTITY DISCOUNT

S. Khosravi and S.H. Mirmohammadi[*, †]
Department of Industrial and systems Engineering, Isfahan University of Technology, Isfahan, Iran

ABSTRACT

Dynamic lot sizing problem is one of the significant problem in industrial units and it has been considered by many researchers. Considering the quantity discount in purchasing cost is one of the important and practical assumptions in the field of inventory control models and it has been less focused in terms of stochastic version of dynamic lot sizing problem. In this paper, stochastic dynamic lot sizing problem with considering the quantity discount is defined and formulated. Since the considered model is mixed integer non-linear programming, a piecewise linear approximation is also presented. In order to solve the mixed integer non-linear programming, a branch and bound algorithm are presented. Each node in the branch and bound algorithm is also MINLP which is solved based on dynamic programming framework. In each stage in this dynamic programming algorithm, there is a sub-problem which can be solved with lagrangian relaxation method. The numeric results found in this study indicate that the proposed algorithm solve the problem faster than the mathematical solution using the commercial software GAMS. Moreover, the proposed algorithm for the two discount levels are also compared with the approximate solution in mentioned software. The results indicate that our algorithm up to 12 periods not only can reach to the exact solution, it consumes less time in contrast to the approximate model.

Keywords: dynamic lot sizing problem; total quantity discount; branch and bound algorithm; dynamic programming; lagrangian relaxation method.

Received: 20 September 2015; Accepted: 8 November 2015

1. INTRODUCTION

One of the major and basic responsibilities in the industrial units is production planning and

---

[*]Corresponding author: Department of industrial and systems Engineering, Isfahan University of Technology, Iran, P.B. BOX: 8415683111
[†]E-mail address: h_mirmohammadi@cc.iut.ac.ir

inventory control. The issue of inventorying material and planning for high quality production with favorable volume at suitable time and reasonable price are of the major concerns of the managers. Economic order quantity models or lot sizing has been developed to achieve this goal.

Economic order quantity determines how much and when a special product should be ordered so that the system costs, which often include holding, ordering and purchasing costs, are minimized [1]. "Dynamic lot sizing programming" refers to those issues where planning horizon is limited and discrete, or in better words, is assumed periodically and the demand is different from one period to another [1]. Many elements affect the variety of lot sizing models, and by definition, in the area of production planning and inventory control, different specifications and assumptions are considered for the model. Among these specifications type of demand, capacity constraints, the number of items, planning horizon and the purchase cost can be noted.

Discount on good purchase has often been raised in deterministic issues. Callerman and Whybark [2] presented a Mixed-Integer Programming (MIP) model for ordering problem with quantity discount through which optimal ordering policy is obtained by binary decision variables. On the issue of determining deterministic lot sizing and considering discount, Chung et al. [3] proved that there is an optimal policy that order quantity between any two consecutive re-ordering points, except for that last order, is equal to one of the discount levels. Using this feature, an algorithm based on dynamic programming algorithm was presented that solve the problem more efficiently than Callerman and Whybark's algorithm. Mirmohammadi et al. [4] presented a branch and bound algorithm for determining the quantity of orders, in deterministic single item cases while considering discount that is more efficient in solving large-scale problems (many periods and high discount levels) compared to previous methods. Goossens et al. [5] demonstrated that there is no polynomial algorithm to solve multi item lot sizing problem considering total quantity discount. In other words, this problem known as TQD is in NP-hard class.

There are two approaches to control unfilled demand in stochastic lot sizing models. Standard approach is introducing the penalty cost for backlogged sales in the objective function. In some cases, calculating this parameter, if not impossible, is too difficult that leads to the use of technical performance standards. The second approach is using service level constraints. The decision makers determine the level of satisfaction with these standards. In the literature of the issue, various performance standards are considered which the most important of them are α, β, γ service levels [6].

The first studies in the field of random demand and considering it in lot-sizing problems was carried out by Silver in 1978 [7]. Silver offered a heuristic three-stage method to determine lot-size with random demand. Bookbinder and Tan [8] modeled the stochastic lot sizing problem in a single-stage state with regard to α service level constraint. In order to control the randomness of demand over time, according to the conditions of inventory and production systems, three strategies have been identified: dynamic uncertainty strategy, static-dynamic uncertainty strategy, static uncertainty strategy. They showed that the mathematical structure of stochastic problem with α service level and static uncertainty strategy is the equivalent to a deterministic lot-size model and deterministic lot sizing problem solving methods can be used to solve the stochastic version. Vargas [9] presented an optimal algorithm for solving the stochastic un-capacitated lot sizing problem. This is called as stochastic version of Wagner–Whitin problem. Sox in [10] dealt with optimal solving of stochastic dynamic lot sizing problem by considering non-stationary purchase cost. Tempelmeier [11] has reviewed the mathematical models of stochastic lot sizing problems and developed a model with static-dynamic strategy considering fill rate β in whose solution inventory on hand is used instead of net inventory. Vargas and Mitters [12] developed the heuristic PDLA algorithm to solve the stochastic single item un-capacitated lot sizing problem in single-stage state by considering penalty cost for unfilled demand in a rolling planning horizon. This algorithm is an extension of optimal algorithm of shortest path problem in static uncertainty strategy.

Quantity discount has newly been studied in stochastic dynamic lot sizing models and few articles have been published in this regard. Hajji et al. studied the quantity discount in single-period model (the newsboy problem) in 2007 with the random initial inventory. For this problem, the optimum quantity of order is determined to maximize profit and the problem is rewritten with random demand and inventory variables in normal mode. In Kang and Lee [14], single item dynamic lot sizing problem with random demand by considering the total quantity discount in supplier selection field has been investigated, a heuristic method based on Dynamic Programming (HDP) has also been developed to solve the problem.

The remainder of the paper has been organized as follows. In Section 2, the problem is defined and beside two mathematical non-linear models of the problem, a piecewise linear model of the problem is presented. Section 3 presents the solution approach of the problem which is based on decomposing the problem in four levels. In each level, a proper approach is applied to handle the sub-problem of the level. In Section 4 some test problems randomly generated are solved by two approaches to evaluate their efficiency relatively. Concluding remarks and results are appeared in Section 5.

2. PROBLEM DEFFINITION AND FORMULATION

In basic models of stochastic dynamic lot sizing problem, it is assumed that the unit price of the ordered items will not change by the quantity of each order. In this paper, a model is investigated in which unit item price depends on the quantity of each order. This means that retailers and product suppliers of commodity offer that if the order quantity x reaches a certain value q, they are willing to sell total value of x to a price lower than c to the buyer. At this point, the newly announced price includes the total value of order x. This price structure is called All-unit discount. This discount cost structure can be defined as non-stationary for the case that the purchase price is non-stationary. In other words, discounts policy is different at any period compared with the others (both in term of price and discount levels). In this study, it is assumed that the number of discount levels K is the same for all periods, but without this assumption the presented model will still be valid. In the intended problem, the demand is considered for one item, and the time horizon is finite. Ordering cost in each period is considered only in case of ordering and resources are unlimited. Demand is assumed to be random and continuous. Shortage is allowed in form of backlogging and the amount of shortage is controlled as shortage penalty cost in the objective function. Demand is random and its density function is known and in any period is independent of other periods. Shortage, holding and ordering costs can vary from one period to another. The goal is to minimize the expected cost of holding cost, shortage cost, ordering cost and purchase cost in total planning horizon. At the beginning of the planning horizon, time and amount of ordering is determined for the entire planning horizon. In this study, the issue Stochastic Single item Discounted Lot Sizing Problem is expressed as with the abbreviation SSDLSP$_{\pi}$.

2.1. Mixed integer nonlinear modes

In this section we formulated the problem in two different ways which lead to two different mixed integer nonlinear models. The following notation is used in the mathematical formulation of the problem:

T	Number of periods in planning horizen
K	Number or discount level

and for period t , $t = 1,2,\ldots,T$,

A_t	Ordering cost
h_t	Holding cost
π_t	Backorder penalty cost
M	A sufficeintly large nublmer
D_t	Demand
x_t	Ordering quantity
X_t	Cumulative order quantity through periods 0 to t $(X_t = \sum_{j=1}^{t} x_j)$
Y_t	Cumulative demand through period0 to t $(Y_t = \sum_{j=1}^{t} D_j)$
$f_{Y_t}(y_t)$	p.d.f of Y_t
$F_{Y_t}(y_t)$	c.d.f of Y_t
$L_t(X_t)$	Total expected holding and penalty costs incurred at the end of period t
s_t	A binary variable which is 1 if an ordering occuredin period t, 0 otherwise
q_{tk}	The minimum acceptable quantity to deserve for discount level k in period t
c_{tk}	Unit Purchasing cost in period t and in discount level k
u_{tk}	A binary variable which is 1 if an order performed in period t in discount level k , 0 otherwise

The problem can be formulated as follows.

$$Min\, E\{c\} = \sum_{t=1}^{T} A_t s_t + L_t(X_t) + c_t(X_t - X_{t-1}) \tag{1}$$

s.t

$$X_{t-1} \le X_t \quad , \ t = 1,\dots,T \tag{2}$$

$$X_t - X_{t-1} \le M\, s_t \quad , \ t = 1,\dots,T \tag{3}$$

$$\sum_{k=1}^{K} u_{tk} = 1 \ , \ t = 1,\dots,T \tag{4}$$

$$\sum_{k=1}^{K} u_{tk}\, c_{tk} = c_t \quad , \ t = 1,\dots,T \tag{5}$$

$$X_t - X_{t-1} \le (q_{t,k+1} - 1) + M(1 - u_{tk}) \quad , \ t = 1,\dots,T \ , \ k = 1,\dots,K \tag{6}$$

$$q_{t,k} + M(u_{tk} - 1) \le X_t - X_{t-1} \quad , \ t = 1,\dots,T \ , \ k = 1,\dots,K \tag{7}$$

$$u_{tk}, s_t \in \{0,1\} \quad , \ t = 1,\dots,T \ , \ k = 1,\dots,K \tag{8}$$

$$c_t, X_t \ge 0 \quad , \ t = 1,\dots,T \tag{9}$$

This model is an extension of Sox [10] model. Because of existence of quantity discount, in this model the unit purchase price in period *t* (c_t) is considered as a decision variable and the expression of the equation (1) is considered as a non-linear expression. This equation represents the expected holding, shortage, ordering, and purchase costs in the total planning horizon. Constraint (2) states that the order amount in period *t* must at least be equal to cumulative order value in the previous period. Constraint (3) is presented to set correct amount to ordering variable (s_t). In this equation, the value of ordering can only be greater than zero when ordering variable gets value one and its cost is considered in the objective function. Constraint (4) shows that the order quantity in each period belongs only to one of the levels. Constraint (5) specifies purchase cost in each period according to the level set for the order. Constraints (6) and (7) fulfil the quantity discount limits in ordering in each period. In these constraints, if an order is given at discount level *k* and in period *t*, the order quantity is limited between $q_{t,k}$ and $q_{t,k+1}$.Otherwise, the constraint for period *t* and level discount *k* is relaxed using the large number *M*. Generally, $q_{t_1} = 0$ and $q_{tk+1} = +\infty$ is assumed in the models and discounts. In the presented model, $L_t(X_t)$ is as the total function of expected shortage and holding costs in accordance with [10] is defined as follows:

$$L_t(X_t) = \begin{cases} h_t(X_t - E[Y_t]) + (\pi_t + h_t)\int_{X_t}^{\infty} (y - X_t)\, f_t(y)dy & if \ X_t \ge 0 \\ \pi_t(-X_t + E[Y_t]) & if \ X_t < 0 \end{cases} \tag{10}$$

If the initial inventory is negative and until period *t* the amount of inventory has not become positive then X_t is negative. In this case, period *t* does not incur any holding cost while the shortage cost is equal to the shortage cost of the amount backlogged till period *t* $(-X_t + E[Y_t])$.

Since the number of zero-one variables has an important effect on computational time, we try to increase the efficiency of the model via reducing the number of zero-one variables in the second model. Furthermore, the purchase cost in objective function has rewritten in linear form.

$$Min\, E\{c\} = \sum_{t=1}^{T} [A_t (\sum_{k+1}^{K} u_{tk}) + L_t(X_t) + \sum_{k+1}^{K} (l_{tk}\, c_{tk})] \tag{11}$$

$$\text{s.t } \sum_{k+1}^{K} l_{tk} = X_t - X_{t-1} \quad t = 1,\ldots,T \tag{12}$$

$$\sum_{k+1}^{K} u_{tk} \le 1 \quad t = 1,\ldots,T \tag{13}$$

$$q_{tk}\, u_{tk} \le l_{tk} \quad t = 1,\ldots,T \quad k = 1,\ldots,K \tag{14}$$

$$l_{tk} \le (q_{tk+1} - 1) u_{tk} \quad t = 1,\ldots,T \quad k = 1,\ldots,K \tag{15}$$

$$u_{tk} \in \{0,1\} \quad t = 1,\ldots,T \quad k = 1,\ldots,K \tag{16}$$

$$X_t \ge 0 \quad t = 1,\ldots,T \tag{17}$$

$$l_{tk} \ge 0 \quad t = 1,\ldots,T \quad k = 1,\ldots,K \tag{18}$$

In this model, ordering cost is determined based on the variable determining level of discount (u_{tk}), and variables s_t, $t = 1,\ldots,T$ are omitted from the model. In this model variable l_{tk} is the amount ordered in discount level *k* at period *t*. In constraint (12), the order size of period *t* is calculated via sum of l_{tk} on all discount levels. Constraint (13) forces ordering to be occurred at most from one of discount levels. Constraints (14) and (15) are set to determine the allowed limits of valuing to l_{tk}.

The first model is more representative than the second one because of simplicity but, in our experimental computations, we adopted the second model due to its time efficiency.

2.2 Piecewise linear approximation model

In the objective function of the previously presented models, the term $L_t(X_t)$ is nonlinear and makes the whole models nonlinear. To reach a solution with a controllable error, we estimate $L_t(X_t)$ by linear approximation and present a linear but approximate model.

$L_t(X_t)$ can be written as follows:

$$L_t(X_t) = h_t\, E[I_t^+] + \pi_t\, E[I_t^-] \tag{19}$$

In equation (19), I_t^+ is positive inventory at period *t* and I_t^- is negative inventory or backlog in period *t*, i.e. $I_t^+ = [X_t - Y_t]^+$ and $I_t^- = [Y_t - X_t]^+$. Therefore, we have $E[I_t^-] = \int_{X_t}^{\infty} (y - X_t) f_{Y_t}(y)$ and $E[I_t^+] = \int_{0}^{X_t} (X_t - y) f_{Y_t}(y)$. $E[I_t^-]$ is named the first order loss function and $E[I_t^+]$ as its the complementary function in the litrature and they can be written on X_t as follows.

$$E[I_t^-] = G_{Y_t}^1(X_t) \tag{20}$$

$$E[I_t^+] = X_t - \mu_t + G^1_{Y_t}(X_t) \tag{21}$$

In equations (20) and (21), $G^1_{Y_t}(X_t)$ is considered as nonlinear function for each period *t*, and for the particular case of normal distribution is defined as follows based on reference [15]:

$$G^1_{Y_t}(X_t) = \sigma_t \left[\varphi\left(\frac{X_t - \mu_t}{\sigma_t}\right) - \left(1 - \phi\left(\frac{X_t - \mu_t}{\sigma_t}\right)\right)\left(\frac{X_t - \mu_t}{\sigma_t}\right) \right] \tag{22}$$

In equation (22), Y_t is the random variable of cumulative demand until period *t* with normal distribution with mean μ_t and standard deviation σ_t and $\varphi(z)$ is the cumulative distribution function of the standard normal distribution.

Consider B points on the X_t- axis in period *t* named e_{bt} , $b = 1,\ldots,B$, that is the value of X_t in b^{th} point in period *t*. Approximation function with B linear pieces presenting total expected holding and shortage costs, is as follows (see [6]):

$$L_t(X_t) = h_t \left[\Delta^0_{I_t^+} + \sum_{b=1}^{B} \Delta^b_{I_t^+} w_{bt} \right] + \pi_t \left[\Delta^0_{I_t^-} + \sum_{b=1}^{B} \Delta^b_{I_t^-} w_{bt} \right] \tag{23}$$

In equation (23) the slope of each piece is defined as follows:

$$\Delta^b_{I_t^+} = \frac{\left[e_{bt} + G^1_{Y_t}(e_{bt})\right] - \left[e_{b-1,t} + G^1_{Y_t}(e_{b-1,t})\right]}{e_{bt} - e_{b-1,t}} \tag{24}$$

$$\Delta^b_{I_t^-} = \frac{G^1_{Y_t}(e_{bt}) - G^1_{Y_t}(e_{b-1,t})}{e_{bt} - e_{b-1,t}} \tag{25}$$

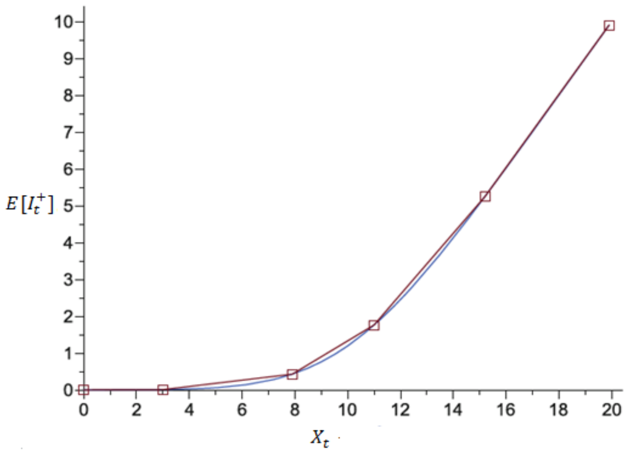


Figure 1. The piecewise approximation of average on-hand inventory

$\Delta^0_{I_t^+}$ and $\Delta^0_{I_t^-}$ are the average inventory and average shortage at the primary point e_{0t}, respectively. w_{bt} is the decision variable defined as the cumulative order amount through the b^th^ interval such that $w_{bt} \le e_{bt} - e_{b-1,t}$. Fig. 1 represent the linear approximation of $E[I_t^+]$ in which $\mu_t = 10$ and $\sigma_t = 10$. In this figure $E[I_t^+]$ has been approximated via 5 linear pieces (*B*=5). Since the expected functions are convex, the slopes $\Delta_{I_t^+}^b$ and $\Delta_{I_t^-}^b$ are increasing on *b*, $b = 1,\ldots,B$. Therefore, due to the minimizing the objective function of the model, w_{bt} become positive only when the previous variable, $w_{b-1,t}$ reaches its maximum value, i.e. $w_{b-1,t} = e_{b-1,t} - e_{b-2,t}$. Therefore, we can set $X_t = \sum_{b=1}^{B} w_{bt}$ in the mathematical model without adding any axillary constraints. So piecewise linear approximation mathematical model of SSDLSP$_\pi$ is as follows:

$$Min\, E\{c\} = \sum_{t=1}^{T} [A_t (\sum_{k=1}^{K} u_{tk}) + h_t \left[\Delta_{I_t^+}^0 + \sum_{b=1}^{B} \Delta_{I_t^+}^b w_{bt} \right] + \pi_t \left[\Delta_{I_t^-}^0 + \sum_{b=1}^{B} \Delta_{I_t^-}^b w_{bt} \right] + \sum_{k=1}^{K} (l_{tk}\, c_{tk})] \tag{26}$$

s.t.

$$\sum_{k=1}^{K} l_{tk} = \sum_{b=1}^{B} w_{bt} - \sum_{b=1}^{B} w_{b,t-1} \quad t = 1,\ldots,T \tag{27}$$

$$\sum_{k=1}^{K} u_{tk} \le 1 \quad t = 1,\ldots,T \tag{28}$$

$$q_{tk}\, u_{tk} \le l_{tk} \quad t = 1,\ldots,T \quad k = 1,\ldots,K \tag{29}$$

$$l_{tk} \le (q_{tk+1} - 1) u_{tk} \quad t = 1,\ldots,T \quad k = 1,\ldots,K \tag{30}$$

$$w_{bt} \le e_{bt} - e_{b-1,t} \quad t = 1,\ldots,T \quad b = 1,\ldots,B \tag{31}$$

$$u_{tk} \in \{0,1\} \quad t = 1,\ldots,T \quad k = 1,\ldots,K \tag{32}$$

$$w_{bt} \ge 0 \quad t = 1,\ldots,T \quad b = 1,\ldots,B \tag{33}$$

$$l_{tk} \ge 0 \quad t = 1,\ldots,T \quad k = 1,\ldots,K \tag{34}$$

3. SOLUTION APPROACH

If Sox's model [10] is considered as a base model with specific solution, the proposed model SSDLSP$_\pi$ is more complex than the basic model in two ways. They are determining the optimal discount levels in each period (determining the optimal amount of variables u_{tk}) and determining the optimal order quantity considering upper and lower limits of permitted ordering. Our strategy to solve the problem is based on decomposition techniques. In this paper, the problem SSDLSP$_\pi$ is solved by branch and bound method. In each node of this algorithm a sub-problem called P_1, is solved by dynamic programming approach. At each stage of this algorithm a sub-problem called P_2 is raised which is solved by a branch and bound method. In each node of the second branch and bound algorithm, sub-problem P_3 is solved by Lagrange relaxation method. In the remainder of this section, the solution approach is presented at four levels. In the first level, sub-problem P_1 is presented and branch and bound algorithm is described. In the second level, sub-problem P_2 is defined and solution of the sub-problem P_1 is presented. In the third level, the solution to sub-problem P_2 with the definition of P_3 is discussed and at the last level, the solution to sub-problem P_3 is described.

3.1. First level

In this section, we first define the sub-problem P_1 and then the solution of the problem with branch and bound method is provided.

3.1.1 Definition of sub-problem P_1

In this sub-problems, it is assumed that the discounting is permitted only for a set of periods, call *D*, and for other periods, call *R*, purchase price is fixed to cheapest case (highest discounts level) with no limit in ordering. Thus, the periods are considered in two sets *R* and *D*. The collection of these two sets includes the complete planning horizon. In other words, the constraints (4) to (8) are applied only for the periods of *D*. In each node of the branch and bound algorithm the discount level for each period of *D* is specified, i.e. the variable u_{tk} in period *t*, $t \in D$, is 1 only for a specific discount level, say m_t, and for other discount levels, $k \neq m_t$, it is zero. Consequently, constraints (6) and (7) change into $q_{tm_t} \le X_t - X_{t-1} \le q_{t,m_{t+1}}$.

For simplicity in model *P_1*, the allowed upper and lower limits for ordering in period *t* are shown by u_t and l_t, respectively. The unit purchase cost c_t is defined as follows:

$$c_t = \begin{cases} c_{tm_t} & t \in D \\ c_{tK} & t \in R \end{cases} \tag{35}$$

The mathematical model of the sub-problem P_1 is as follows:

$$\text{Min } \mathrm{E}\{c_{P_1}\} = \sum_{t=1}^{T} A_t s_t + L_t(X_t) + c_t(X_t - X_{t-1}) \tag{36}$$

$$\text{s.t } \mathrm{X_t - X_{t-1} \le u_t} \quad \forall t \in D \tag{37}$$

$$\mathrm{X_t - X_{t-1} \ge l_t} \quad \forall t \in D \tag{38}$$

$$\mathrm{X_{t-1} \le X_t} \quad t = 1,\ldots,T \tag{39}$$

$$\mathrm{X_t - X_{t-1}} \le M\mathrm{s_t} \quad t = 1,\ldots,T \tag{40}$$

$$s_t \in \{0,1\} \quad t = 1,\ldots,T \tag{41}$$

$$X_t \ge 0 \quad t = 1,\ldots,T \tag{42}$$

In this model, the objective function is defined as in [10] and can be rewritten as follows:

$$\text{Min E}\{\mathrm{c}_{\mathrm{P}_1}\} = \sum_{t=1}^{T} A_t s_t + L_t(X_t) + (c_t - c_{t+1})X_t \tag{43}$$

Property 1: If we set *D* as an empty set, the optimal solution of the problem P_1 is a lower bound for the problem SSDLSP$_\pi$.

Emptiness of set *D* means that the problem has no limits for ordering with regard to the discount policies. Thus, by reducing the number of constraints, the solution space will get greater. On the other hand, the best purchase price is considered for all periods. So the obtained solution will be the best possible solution for the expected total cost.

Property 2: The optimal solution of sub-problem P_1 is an upper bound for the problem SSDLSP$_\pi$.

The discounted cost structure and related constraints are incorporated only for the periods of *D* and the purchase cost of other periods is set to the lowest case. Hence, it is obvious that in this circumstance the obtained solution is a lower bound for the more restricted general case of SSDLSP$_\pi$.

The main benefit of applying a B&B algorithm for solving SSDLSP$_\pi$ is that it enumerates all possible circumstances of variable u_{tk}, $t = 1,\ldots,T$, $k = 1,\ldots,K$, and solving the related sub-problem. Therefore, the optimal solution to the problem SSDLSP$_\pi$ is obtained by this approach. In a complete enumeration, there is *K* quantities for discount level in each period *t*, $t = 1,\ldots,T$. Hence, there are K^T possible quantities for u_{tk} variable. The presented B&B approach enumerate these cases implicitly.

3.1.1 The first level branch and bound algorithm

In this section, the steps of the branch and bound algorithm which breaks down SSDLSP$_\pi$ to P_1 are presented.

To obtain an initial solution for the problem and use it as an lower bound from the beginning of the B&B algorithm at the initial level, it is assumed that there are no ordering constraints and the unit purchase price in all periods is the lowest possible value (maximum discount). The solution space with this assumption is much larger than the original problem and its objective function value is a lower bound for the problem. The problem with the mentioned assumption is in fact, the problem introduced by Sox [10].

In Branching step of the algorithm, a period is selected and it is inserted to set *D*. The strategy of selecting the period depends on the value of orders in the initial solution which is in the root node. In other words, the periods with positive ordering value in the initial solution are in priority of branching. More precisely, at the root node, a list of prioritizing with the mentioned criterion is determined for branching and branching happens according to this list for different periods. For each node, as a parent node, *K* nodes , as children node, are generated by adding the related constraints for each discount level k, $k = 1,\ldots,K$. More precisely, a period, say period *t*, is selected from the mentioned list and for each discount level k, $k = 1,\ldots,K$, *K* nodes in period *t* is generating such that in each node a sub problem P_1 with the upper and lower limit constraints , corresponding to the discount level k, is added for the ordering value on period *t*. Branching continues from the active node that has the best lower bound. If two nodes with equal lower bound found, branching is done in the node that has more depth in the search tree.

After obtaining lower bound of each node by solving P_1, the values of ordering are calculated with their related prices I accordance with the discount cost structure to satisfy the discount level constraints of that period. In this way, an upper bound is obtained for the problem.

In this algorithm, nodes are fathomed with the following rules:

1. If the upper and lower bounds are equal in a node, it means that the resulting solution is feasible and branching will not better the answer
2. If the lower bound of a node is greater than the best so far upper bound.

It should be noted that if the node is at the lowest level of the tree, the first rule is true about it and the node is fathomed.

At the end of the algorithm, all nodes are fathomed and the node with the lowest objective function whose is upper and lower limits are equal is the optimal solution.

3.2. Second level

To be an optimal algorithm, the B&B algorithm must solve the sub problem P_1 in each node optimally. For this purpose, a dynamic programming algorithm is applied that in each stage a sub-problem, named P_2, is raised.

3.2.1 Definition of sub problem P_2 (s, t)

Consider a sub problem P_1 raised in a node of B&B algorithm. Solving this sub problem by our dynamic programming approach lead to be raised a sub problem $P_2(s,t)$ in each stage of the DP algorithm. In $P_2(s,t)$ the goal is to find the minimum expected cost of holding, ordering, purchasing and shortage from period s to the end of period t, assuming that the ordering occurs only in the period s and if there are any discount constraints, in the period t_1 to t_m with following conditions:

1. Period s and $t+1$ are two periods of sub problem P_1 which do not have any ordering constraint. (i.e. $s,t+1 \in R$)
2. $M = \{t_1,..t_m\}$ ($m \le t-s$) is the set of all the periods between s and t that have constraints in ordering. ($s = t_0 < t_1,\dots,t_m < t_{m+1} = t+1$, $M \subset D$)

Mathematical model of sub problem $P_2(s,t)$ is as follows:

$$Min\ P_{st} = A_s + \sum_{i=1}^{m} A_{t_i} s_{t_i} + \sum_{i=0}^{m} g_{t_i,t_{i+1}-1}(X_{t_i}) \tag{44}$$

$$\text{s.t } X_{t_i} - X_{t_i-1} \le u_{t_i} s_{t_i} \quad \forall i = 1,\dots,m \tag{45}$$

$$X_{t_i} - X_{t_i-1} \ge l_{t_i} \quad \forall i = 1,\dots,m \tag{46}$$

$$s_{t_i} \in \{0,1\} \quad \forall i = 1,\dots,m \tag{47}$$

$$X_{t_i} \ge 0 \quad \forall i = 0,\dots,m \tag{48}$$

In equation (44), function $g_{i,j}(X_i)$ expresses the expected total holding, shortage and purchase costs from period *i* to the end of period *j* when cumulative order from period *i* to the end of period *j* is equal to the fixed amount X_i. So $g_{i,j}(X_i)$ is as follows.

$$g_{i,j}(X_i) = \begin{cases} (c_i - c_{j+1})X_i + \sum_{t=i}^{j} L_t(X_i) & 0 < i \le j \\ -c_{j+1}X_0 + \sum_{t=i}^{j} L_t(X_0) & i = 0 \end{cases} \tag{49}$$

In addition, total ordering cost is equal to the sum of ordering cost in period *s* and if any, the ordering costs in t_1 and t_m. Equation (45) is a combination of the two constraints (37), (40) and states that in the middle periods, if there is an order, its amount should not exceed the upper limit of related discount level value of that period. It is notable that for the periods that lower limit of discount level for them (l_{t_i}) is greater than zero, the related variable s_{t_i} will be one. Thus, that part of ordering cost which is variable is only considered for a subset of $D_z \subset M$ whose lower permissible limit of the discount level for them is zero since for periods whose ordering amount are located on the first discount level we have $l_{t_i} = 0$. Base on what is stated above, we can rewrite the objective function of the model as equation (50).

$$Min\ P_{st} = A_s + \sum_{t_i \notin D_z, t_i \in M} A_{t_i} + \sum_{t_i \in D_z} A_{t_i} s_{t_i} + \sum_{i=0}^{m} g_{t_i, t_{i+1}-1}(X_{t_i}) \tag{50}$$

3.2.2 Dynamic programming algorithm

In this section, a forward dynamic programming is presented which solves the sub problem P_1 and in each its stage a sub problem $P_2(s,t)$ must be solved. As mentioned earlier, a sub-problem $P_2(s,t)$ determines the optimal ordering policy from period *s* to the end of period *t* named by X_{st}^{opt}. In fact, X_{st}^{opt} is a vector of optimal orders amount from period *s* to period *t*, *i.e.* $X_{st}^{opt} = (X_s^*, X_{s+1}^*, .., X_t^*)$, which minimizes the expected total cost during mentioned periods under the mentioned assumption for sub problem $P_2(s,t)$. Let the minimum expected cost, mentioned above, is denoted by $P_{st}^{opt}(X_{st}^{opt})$ and suppose t_w, $w = 1,..,W$, be the periods of set *R* where we have $s_{t_w} = 1$. In this case, the objective function of the sub problem can be rewritten as the sum of several objective functions of the sub-problem $P_2(s,t)$:

$$E\{c_{p_1}\} = \sum_{w=1}^{W} P_{t_w, t_{w+1}-1}(X_{t_w, t_{w+1}-1}^{opt}) \tag{51}$$

In equation (51) $P_{t_w, t_{w+1}-1}(X_{t_w, t_{w+1}-1}^{opt})$ shows the optimal value of objective function of sub-problem P_2 from period t_w to period $t_{w+1} - 1$, when the optimal order quantity is equal to

$X^{opt}_{t_w,t_{w+1}-1}$. In this equation $t_{w+1} = T+1$. This equation indicates that sub problem P_1 is separable to sub problems $P_2(t_w, t_{w+1}-1)$.

The proposed dynamic programming algorithm has the following elements:

Stage: calculation of the minimum cost to period t ($t \in R$).

State: Period s ($s \in R$) as the last ordering period before the period t.

Recursive relationship: f_{st} is the minimum cost associated with the ordering policy form period s to t when the last order is occurred among the periods of R in period s and next order in this set occurs at period $t+1$, we have

$$f_{st} = P_{st}(X^{opt}_{st}) + \min_{k<s,k\in R}\{f_{k,s-1} : X^{opt}_{k,s-1}(s-1) \le X^{opt}_{st}(s)\} \tag{52}$$

If $\{X^{opt}_{k,s-1} \mid X^{opt}_{k,s-1} \le X^{opt}_{st}\} = \phi$ then $f_{st} = \infty$. This situation means that in the optimal policy there is no policy by which it is possible to reach s, and then go to $t+1$, because this path violates Constraint (39).

The pseudo code for the proposed algorithm is as follows

For $t = 0,\dots,T$ *and* $t+1 \in R$

For $s = 0,\dots t$ *and* $s \in R$

Solve $P_2(s,t)$ *and compute* X^{opt}_{st} *and* $P_{st}(X^{opt}_{st})$

If $s = 0$, $MIN = 0$

Else

If $\{k \mid k < s, k \in R : X^{opt}_{k,s-1}(s-1) \le X^{opt}_{st}(s)\} = \phi$, $MIN = \infty$

Else

$MIN = \min_{k<s,k\in R}\{f_{k,s-1} : X^{opt}_{k,s-1}(s-1) \le X^{opt}_{st}(s)\}$

$f_{st} = P_{st}(X^{opt}_{st}) + MIN$

$t = T$

While $t > 1$

if $f_{jt} = \min_{k<t,k\in R}\{f_{kt}\}$ $s = j$

for $j = s,\dots t$, $X_j = X^{opt}_{st}(j)$

$t = (s-1)$

3.3 Third level

If the optimal values of binary ordering variables are determined in sub problem $P_2(s,t)$, then the problem will become a non-linear non-integer programming that convex optimization methods can be used for optimal solving of the model.

3.3.1 Definition of sub problem P_3

As previously mentioned, the collection D_z is a series of periods between s and t where the lower limit of order is zero. So in these periods ordering may be issued or not. A sub-problem P_3 is the same as sub-problem P_2 where it is decided beforehand for ordering in the periods relating to set D_z. In this case; periods of D_z are divided into three categories. The first set is D_A where the cost of ordering for its member periods is determined accordance with the terms defined in the sub-problem P_2 and considered in the objective function of the problem (whether there is an order in those periods or not), in other words $s_{t_i} = 1$. The second set is D_B where the cost of ordering for its periods is considered as zero, and the order is considered free in those periods. In other words $s_{t_i} = 0\ (t_i \in D_B)$. The third set is D_C that is assumed not to perform any orders in those periods, in other words $s_{t_i} = 0\ (t_i \in D_c)$. Community of these three sets composes D_z set.

Then for ease in model P_3 the cost A_t is defined as follows:

$$\overline{A}_t = \begin{cases} A_t & t \in D_A \\ 0 & t \in D_B \end{cases} \tag{53}$$

With this division for periods of D_z the sub problem P_2 changes into a problem where there are no zero-one variables of ordering. Moreover, ordering occurs only in the period s and in the periods of the series $N = \{t_1,..t_n\}$. This collection is a subset of M where ordering variable for all its periods in the model P_2 is equal to one ($s_{t_i} = 1$). In other words, the period of D_C set has been removed from the collection of t_1 to t_m ($N = M - D_C$).

Mathematical programming model of the problem P_3 is as follows:

$$Min\, PR_{st} = \sum_{i=0}^{n} \overline{A}_t + g_{t_i,t_{i+1}-1}(X_{t_i}) \tag{54}$$

$$\text{s.t } X_{t_i} - X_{t_i-1} \geq u_{t_i} \quad \forall i = 1,\dots,n \tag{55}$$

$$X_{t_i} - X_{t_i-1} \geq l_{t_i} \quad \forall i = 1,\dots,n \tag{56}$$

$$X_{t_i} \geq 0 \quad \forall i = 1,\dots,n \tag{57}$$

In this model, periods are assumed as follows: $(s = t_0 < t_1,\dots,t_n < t_{n+1} = t+1\ ,\ t_1,\dots,t_n \in N)$

Property 3: In P_3, if all periods of D_z are placed in set D_B, the optimal solution for sub problem P_3 would be a lower bound for sub problem $P_2(s,t)$.

Property 4: From any optimal solution of the sub problem P_3, it is possible to reach an upper bound for the sub problem P_2 by modifying the order costs with regard to the solution obtained.

Property 5: By complete enumeration of different values of variable s_{t_i} $(t_i \in D_z)$ and solving the related sub problems, the optimal solution of P_2 will be determined.

3.3.2 The third level branch and bound algorithm

In this section, in an algorithm similar to the mentioned branch and bound algorithm, a tree search based on the branch and bound approach is presented by which the s_{t_i}'s are determined are determined in sub problem P_2 and optimal value of ordering are obtained.

The initial solution for the root node of the branch and bound algorithm is considered as follows. In P_2, variable s_{t_i} has appeared as a complicating variable. If we assume that in each period, one can freely order and there is no charge for ordering, then the lower bound of the problem is specified. Therefore, at the root node, ordering cost will be considered for none of the periods of D_Z and all the periods of D_Z are placed in D_B.

At the branching step of the algorithm, parent node is branched for a period from among the periods of D_Z. This branching generates two child nodes for a parent node where in one the ordering cost is incurred and ordering is occurred in the mentioned period (the mentioned period is placed in D_A) and the other one, the ordering cost is set to zero and no ordering is occurred in the mentioned period. (the mentioned period is placed in D_C).

The strategy of node selection for branching is as follows. In this algorithm, among the active nodes, the node with the best lower bound is selected for branching. If two nodes have equal lower bounds, branching is done on the node with larger depth in the search tree.

In this algorithm, nodes are fathomed with the following fathoming rules:

If the upper and lower bounds are equal in a node, it means that the resulting solution is feasible and branching would not lead to better solution.

If the lower bound of a node is greater than the upper bound obtained so far.

If for all periods of D_Z the decision of ordering is determined. It means the node is inactive and no further branching is needed.

The optimal solution of the sub problem will be obtained at the end of the algorithm. Among the fathomed nodes, the node with the lowest objective function whose upper and lower limits are equal is the optimal solution.

3.4 Fourth level

The Lagrange relaxation method is used in cases whereby relaxing a number of constraints of the problem leads to a more simple problem. In sub problem P_3, if the permissible lower and upper limits of the order in equation (55) and (56) are relaxed, then the solution of the relaxed problem can easily be obtained by derivative of the objective function. In this section, the Lagrange relaxation method (LR) is presented to solving the sub problem P_3 in three main steps.

3.4.1 Step 1: Solving the relaxed version of P_3 (RPP)

The Lagrange relaxed version of the sub problem P_3 can be stated as follows:

$$Min\ \ RPP_{st} = \sum_{i=0}^{n} A_{t_i} + g_{t_i,t_{i+1}-1}\left(X_{t_i}\right) + \sum_{i=1}^{n}\left[\bar{\lambda}_{ut_i}\left(X_{t_i} - X_{t_{i-1}} - u_{t_i}\right) - \bar{\lambda}_{lt_i}\left(X_{t_i} - X_{t_{i-1}} - l_{t_i}\right)\right] \tag{58}$$

s.t.

$$X_{t_i} \geq 0 \quad \forall i = 1, \cdots, n \tag{59}$$

Lagrange multipliers associated with the upper and lower limits of the ordering constraints are $\bar{\lambda}_{ut_i}$ and $\bar{\lambda}_{lt_i}$, respectively. We assume that $\bar{\lambda}_{ut_0} = \bar{\lambda}_{lt_0} = \bar{\lambda}_{lt_{n+1}} = \bar{\lambda}_{ut_{n+1}} = 0$. Therefore, for each period like t_i, there are three variable X_{t_i}, $\bar{\lambda}_{lt_i}$ and $\bar{\lambda}_{ut_i}$. The objective function of *RPP* can be rewritten as follows.

$$Min\ RPP_{st} = \sum_{i=0}^{n} A_{t_i} + f_{t_i}\left(X_{t_i}\right) + l_{t_i} \cdot \bar{\lambda}_{lt_i} - u_{t_i}\bar{\lambda}_{ut_i} \tag{60}$$

s.t.

$$X_{t_i} \geq 0 \quad \forall i = 1, \cdots, n \tag{61}$$

where $f_{t_i}\left(X_{t_i}\right)$ is defined as follows.

$$f_{t_i}\left(X_{t_i}\right) = \left[\left(c_{t_i} + \bar{\lambda}_{ut_i} - \bar{\lambda}_{lt_i}\right) - \left(c_{t_{i+1}} + \bar{\lambda}_{ut_{i+1}} - \bar{\lambda}_{lt_{i+1}}\right)\right]\overline{X}_{t_i} + \sum_{j=t_i}^{t_{i+1}-1} L_j\left(\overline{X}_{t_i}\right) \tag{62}$$

Thus, the relaxed version of the sub problem P_3 is an unconstrained non-linear mathematical programming where its objective function is concave. Hence, its optimal solution can easily be obtained by derivative. The partial derivative of the objective function relative to X_{t_i} variable is as follows.

$$\frac{\partial RPP_{st}\left(X_{t_i}\right)}{\partial X_{t_i}} = \sum_{j=t_i}^{t_{i+1}-1}\left[h_j - (h_j + \pi_j)(1 - F_{y_j}(X_{t_i}))\right] + \left(c_{t_i} + \bar{\lambda}_{ut_i} - \bar{\lambda}_{lt_i}\right) - \left(c_{t_{i+1}} + \bar{\lambda}_{ut_{i+1}} - \bar{\lambda}_{lt_{i+1}}\right) \tag{63}$$

To find the roots of this function a combination of bisection method and false position iteration methods are used.

3.4.2 Step 2: Updating the Lagrange multipliers

The main step in Lagrange relaxation is updating Lagrange multipliers for which in the related literature, various methods have been developed. In these methods, the goal is to maximize the dual problem of the original relaxed model via changing some coefficients of Lagrange multiplier which subsequently leads to minimize original problem the original variables. One of the main and general methods of updating Lagrange multipliers is using sub-gradient method. For the sub-problem of P_3, Lagrange multipliers are updated as follows:

$$\bar{\lambda}_{ut}^{(\upsilon+1)} = \max\left\{0, \bar{\lambda}_{ut}^{(\upsilon)} + t^{(\upsilon)} G_{ut}^{(\upsilon)}\right\} \tag{64}$$

$$\bar{\lambda}_{lt}^{(\upsilon+1)} = \max\left\{0, \bar{\lambda}_{lt}^{(\upsilon)} + t^{(\upsilon)} G_{lt}^{(\upsilon)}\right\} \tag{65}$$

$$t^{(\upsilon)} = \frac{\pi^{(\upsilon)}\left(UB_{best} - RPP_1^{(\upsilon)}\right)}{\sum_{i=1}^{n}\left[\left(G_{lt}{}^{(\upsilon)}\right)^2 + \left(G_{ut}{}^{(\upsilon)}\right)^2\right]} \tag{66}$$

In these equations $G_{ut}{}^{(\upsilon)} = X_t{}^{(\upsilon)} + X_{t-1}{}^{(\upsilon)}$ and $G_{lt}{}^{(\upsilon)} = l_t - X_t{}^{(\upsilon)} + X_{t-1}{}^{(\upsilon)}$. Usually, in the literature we found that $\pi^{(0)} = 2$ and for better convergence $\pi^{(\upsilon)}$ is reduced during iterations of the algorithm. UB_{best} is the best upper bound obtained from the primary problem P_3 that are obtained in a heuristic way by making the solution feasible from the relaxed problem by changing order amount to meet the feasible limits values of constraints. $RPP_{st}{}^{(\upsilon)}$ is the value of the objective function of the relaxed problem in v^{th} step.

3.4.3 Step 3: Stopping criteria

At this step, the predetermined convergence criteria are check to ensure that the obtained solution is sufficiently close to the optimal solution to stop algorithm. Following criteria are the main conditions stated in the literature.

If the difference of the best lower bound (RPP_{best}) with the best upper bound (UB_{best}) in the algorithm is less than an error ε , the algorithm is terminated and the solution is reported as ε -optimal solution.

If the number of occurrences is greater than a specified limit, the algorithm will stop.

If the vector of Lagrange multipliers are sufficiently close in the last iterations ($\left\|\lambda^{(\upsilon+1)} - \lambda^{(\upsilon-1)}\right\| / \left\|\lambda^{(\upsilon)}\right\| \le \varepsilon$), the algorithm will stop.

Overall Lagrange algorithm descried in this section is summarized in form of following pseudo code.

Step 0: Initialization.

Set $\upsilon = 1$,

Initialize dual variables $\lambda^{(\upsilon)} = \lambda$

Set $\varphi_{down}^{(\upsilon-1)} = -\infty$

Step 1: Solution of the relaxed primal problem.

Solve the relaxed primal problem and obtain optimal value of $x^{(\upsilon)}$ and its associated objective function $\varphi^{(\upsilon)}$

Update the lower bound for the objective function of the primal problem,

If $\varphi_{down}^{(\upsilon-1)} < \varphi^{(\upsilon)}$ set $\varphi_{down}^{(\upsilon)} = \varphi^{(\upsilon)}$

Step 2: Multiplier updating.

Update multipliers using sub-gradient method. If possible, update also the objective function upper bound.

Step 3: Convergence checking.

If the stopping criterion is met, the ε -optimal solution is $x^{(\upsilon)} = x^*$ and stop. Otherwise set $\upsilon = \upsilon + 1$ and go to Step 1.

4. COMPUTATIONAL EXPERIENCE

Numerical experiments were carried out with the programming the proposed algorithm in C # in Microsoft Visual Studio 2010. The results of solving the test problems are analyzed for evaluating the performance of the algorithm. Also the results are compared with the results obtained with the results obtained from solving the mixed-integer nonlinear model of SSDLSP$_\pi$ and mixed-integer linear approximation model which have been run in in commercial optimization software GAMS. This comparison is performed based on computational time and accuracy of solution by solving a set of random problems. Among solver in GAMS only solvers BONMIN, KNITRO, ALPHAECP and LINDOGLOBAL are available for MINLP models in which the first-order normal loss function is definable. The only one of them which is able to announce the optimal solution on a small scale is LINDOGLOBAL. Other solvers, even if they have reached the global optimum solution, report it as a local optimal solution. So the basis of comparison of solution methods is to solve the problem in GAMS with the solver LINDOGLOBAL. This solver can also be used in LINGO software.

4.1 Experimental design

In generating test problems, each of the input data is a controlling factor. Among these factors, the impact of two factors T and K are of more importance than other factors in solving the problem. Other inputs have been adjusted experimentally and they have fixed through all test problems. Ordering, holding, and shortage costs and initial inventory have been set in accordance with reference [10]. The expected value of demand, $E(d_t)$, for each period is determined randomly. The standard deviation of demand for each period is assumed as $\sigma(d_t) = 0.2 * E(d_t)$ like what has been done by [10]. Therefore, the expected value and standard deviation of cumulative demand till period *t*, is equal to $\mu_t = \sum_{j=1}^{t} E(d_j)$ and $\sigma_t = \sqrt{\sum_{j=1}^{t} \sigma^2(d_j)}$, respectively. The first level unit price of the unit purchasing price in discounted cost structure in each period (c_{t1}) is determined randomly as shown in Table 1. Also, Table 1 shows the adjusted values of other input parameters which have been randomly generated.

Table 1: Adjusted values of the input parameters

Parameters	c_{t1}	$E(d_t)$	I_0	π_t	h_t	A_t
adjusted value	[0,8]	[20,120]	98	12	0.5	**48**

In setting the discount policy parameters, two factors are have great importance generating test problems. The first one is α_k**,** the proportion of discount level to the average demand by which the minimum ordering amount in k^{th} level in the t^{th} period, q_{tk}, is determined i.e. $q_{tk} = \alpha_k . \mu_t$. The second important parameter is **γ,** the interest rate of discount, which is an indicator to measure the purchase cost saving based on which the purchase cost is obtained with the equation $c_{tk} = (1-\gamma_k).c_{t1}$.

These two factors, similar to the Mirmohammadi et al. [4], have been set in deterministic the lot sizing problem with quantity discount. Their value have been listed in Table 2 for five discount levels.

Table 2: Adjusted value of parameters for discounting policy

K	**2**	**3**	**4**	**5**
α_k	2	3	4	**5**
γ_k	0.05	0.075	0.1	0.125

To consider the cases for more than two random parameters μ_t and c_{t1}, random generation of data for each problem with T & K is repeated five times and the time to solve problems is obtained from the average of five problems.

4.2. Analysis of test results

The average time to solve the problems are recorded from the implementation of the program on the machine with specs Intel (R) core (TM) i7-2600 CPU@3.40 GHz and they have listed in Table 3. In this table, the mean of LG is the solver LINDOGLOBAL and B&B refers to the proposed Branch and bound algorithm. *Rt* shows the average computational time for every five test problems solved with various values of *T* and *K* in seconds. As shown in Table 3, LG solver is unable to solve problems with more than 9 periods in the specified time limit. *N* shows the number of instances of problems (from five instances) which are solved optimally in a time less than 7200 seconds.

Table 3: Run time comparison of B&B and LG

Num		B&B			LG					
	K	2	3	4	2		3		4	
	T	*Rt*	*Rt*	*Rt*	*Rt*	*N*	*Rt*	*N*	*Rt*	*N*
1	3	0.118	0.298	0.319	3.178	5	3.793	5	4.978	5
2	4	0.290	0.375	0.599	7.972	5	9.783	5	13.813	5
3	5	0.417	0.698	0.698	33.860	5	36.115	5	43.965	5
4	6	0.520	0.694	1.895	203	5	203	5	260	4
5	7	0.608	1.358	3.712	956	5	1368	5	4049	3
6	9	0.939	2.846	3.710	6194	3	7081	1	7200	0
7	10	1.412	2.556	7.969	7200	0	7200	0	7200	0
8	11	2.432	5.201	8.996	7200	0	7200	0	7200	0

Fig. 2 shows the computational time of B&B in comparison with solver LG with two discount levels. As it is shown in this figure, the computational time of the B&B algorithm is drastically less than the what is obtained from solver LG. This solver is not able to announce the optimal solution in 9 and 10 periods in a time less than 7200 sec.. This is while the longest computational time of the proposed algorithm is for the case in which K=2 and T=2 with time 2.432 seconds. This indicates the high performance of proposed algorithms.

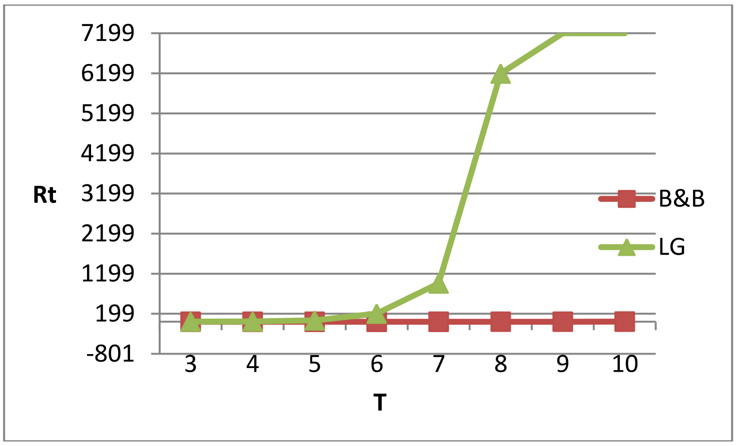


Figure 2. Run time comparison of the solver B&B and LG with two discount level

Since in Fig. 2 changing the values relatively is not tangible, the bottom part of this graph is magnified in Fig. 3.

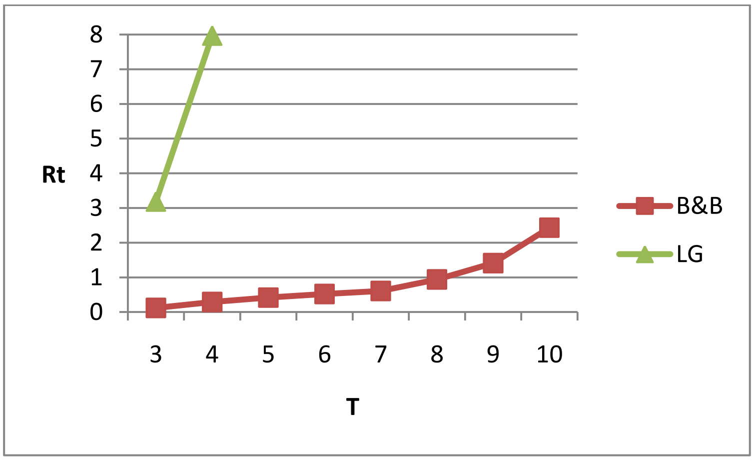


Figure 3. Partial magnified of Figure 1

Two levels discount problem has more practical aspect than other variant of this problem. Then the behavior of the proposed algorithm for two levels of discounts up to 30 periods is compared. One of the main issues in the analysis of the behavior of nonlinear algorithms is comparing them with their linear approximation version. In this respect, the approximation model of the problem was encoded in GAMS software by considering 1000 approximation points in each shortage function in the objective function. The computational results are shown in Table 4. In this table, *Rt* is the average run time for all five instances. APP stands for linear approximation model that runs on GAMS software. E means the relative error of solution of approximation model to *B & B* algorithm. *Num* shows the problem number.

Table 4: Run time comparison of B&B and APP

Num	T	B&B Rt	APP Rt	E
1	4	0.26	0.495	1.1
2	6	0.556	1.016	1.43
3	8	0.736	1.513	0.82
4	10	1.385	2.147	0.88
5	12	2.232	2.918	0.99
6	14	3.833	3.768	0.83
7	16	7.882	4.794	1.36
8	18	17.801	5.936	1.29
9	20	25.24	7.175	1.17
10	22	30.629	10.1	1.2
11	24	31.279	10.905	1
12	26	69.001	13.521	1.44
13	28	71.811	14.43	1.34
14	30	281.79	15.42	1.4

As shown in Fig. 4, the B&B algorithm gets the optimal solution faster than the approximate model (APP). From this point of intersection of the curves in Fig. 4, the priority of user should be specified, if the accuracy of the solution has higher priority, the proposed B&B algorithm should be applied, and if the solution time is important, ignoring the relative error, the approximation model is better.

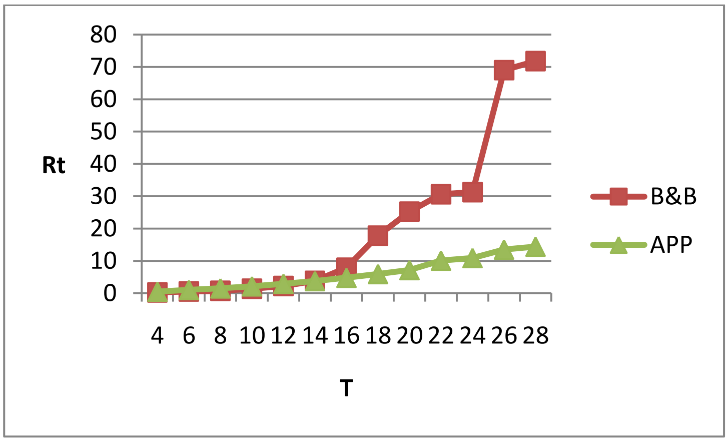


Figure 4. Run time comparison of B&B and APP

5. RESULTS AND CONCLUSIONS

Adding the assumption of possibility of discount in material purchasing to the Sox's model [10] and deffining the stochastic single item lot sizing problem under quantity discount in purchasing (SSDLSPπ) make the problem much more complex from two points of view. Adding several binary variables to the model is the first aspect and the second one is the extra constraints added to the base model. Thus, although the base model has been solved with a dynamic programming algorithm, the developed model of the algorithm has a more complex solution approach. The proposed solution approach is presented at four levels based on a branch and bound algorithm hybridized with a dynamic programming algorithm to obtain the optimal solution of the problem. This approach can be used for any arbitrary distribution of demand, and is faster than LINDOGLOBAL solver in GAMS. Furthermore, to have a more precise evaluation of the presented algorithm in large scale problems, we presented the linear approximate model of the model and we compared the presented algorithm with it. The presented algorithm solves the problem with 30 periods (T=30) optimally in a reasonable time, but, slower than approximate model. The approximate model performs more efficient than B&B algorithm but with a bit of error to the optimal solution. In our experiments, the maximum error of the approximate model is 1.44 percent which seems tolerable regarding the speed of this approach.

In this paper we contained the shortage of product by charging the shortage cost to the objective function of the problem. Since evaluating the shortage cost parameters may be hard in practice, handling the shortage of products via defining the proper customer service level may me more practical and it is left as a future development of the current work.

REFERENCES

1. Karimi B, Fatemi Ghomi S, Wilson J. The capacitated lot sizing problem: a review of models and algorithms, *Omega* 2003; **31**: 365-78.
2. Callerman T, Whybark D. Purchase quantity discounts in an MRP environment, *Proceeding of 8th A nnual Midwest Conference*, 1977.
3. Chung, CS, Chiang DT, Lu CY. An optimal algorithm for the quantity discount problem, *J Oper Manag* 1987; **7**: 165-77.
4. Mirmohammadi SH, Shadrokh S, Kianfar F. An efficient optimal algorithm for the quantity discount problem in material requirement planning, *Comput Oper Res* 2009; **36**: 1780-88.
5. Goossens DR, Maas A, Spieksma FC, Van de Klundert J. Exact algorithms for procurement problems under a total quantity discount structure, *European J Oper Res* 2007; **178**: 603-26.
6. Tempelmeier H. Stochastic lot sizing problems, *Handbook of Stochastic Models and Analysis of Manufacturing System Operations* 2013, Springer, pp. 313-44.
7. Silver E. Inventory control under a probabilistic time-varying, demand pattern, *AIIE Trans* 1978, **10**: 71-9.
8. Bookbinder JH, Tan JY. Strategies for the probabilistic lot-sizing problem with service-level constraints, *Manag Sci* 1988; **34**: 1096-1108.
9. Vargas V. An optimal solution for the stochastic version of the Wagner–Whitin dynamic lot-size model, *European J Oper Res* 2009; **198**: 447-51.
10. Sox CR. Dynamic lot sizing with random demand and non-stationary costs, *Oper Res Letters* 1997; **20**: 155-64.
11. Tempelmeier H. On the stochastic uncapacitated dynamic single-item lotsizing problem with service level constraints, *European J Oper Res* 2007; **181**: 184-94.
12. Vargas V, Metters R. A master production scheduling procedure for stochastic demand and rolling planning horizons, *Int J Product Economics* 2011; **132**: 296-302.

13. Haji M, Haji R, Darabi H. Price discount and stochastic initial inventory in the newsboy problem, *J Indust Syst Eng* 2007; **1**: 130-8.
14. Kang HY, Lee AH. A stochastic lot-sizing model with multi-supplier and quantity discounts, *Int J Product Res* 2013; **51**: 245-63.
15. Rossi R, Kilic OA, Tarim SA. Piecewise linear approximations for the static–dynamic uncertainty strategy in stochastic lot-sizing, *Omega* 2015; **50**: 126-40.