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OPTIMAL CONTROL OF PUMPING STATIONS IN OPEN CHANNELS BY METAHEURISTIC FIREFLY ALGORITHM

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ABSTRACT

Optimum control of upstream pumping station in open channels with given constraint in downstream end is presented in this paper. The upstream control is capable of minimizing water level fluctuations in the channel in which the downstream pumping station causes an undesirable wave. The proposed method combines an unsteady non-uniform flow solver with shock-capturing capability, Fourier series and metaheuristic firefly algorithm. Fourier series is used to estimate the optimum inflow control and firefly algorithm is utilized to determine the unknown coefficients in the series. With a suitable objective function, the procedure generates the optimum inflow hydrograph that can effectively cancel destructive downstream waves. The results have been compared with the results obtained by a variational approach and show satisfactory improvement both in simplicity and the value of objective function.

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KEY WORDS: optimal control; unsteady open channel flow; pumping stations; firefly algorithm

1. INTRODUCTION

Rapidly varied flows usually develop shocks in flow domain which should be treated accordingly in flow simulation models. Development of shocks is due to abrupt changes in flow conditions such as flow rate or water depth. Various well-known shock-capturing

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schemes are available in the literature to deal with shocks [1-6].

In common water conveyance projects, a pumping station at upstream end of the channel supplies water from a river or well and a downstream pumping station pumps the necessary water to the consumers. Unsteady conditions can occur for example by sudden change in downstream water demand. In order to compensate the increased downstream demand, the upstream flow rate supplied by pumping station should be increased accordingly. This discharge increment in both upstream and downstream ends produces undesirable surges which travel downstream and upstream of the channel [7, 8]. Depending on the flow rates, large oscillations in water level cause severe problems such as serious water losses due to water level exceeding the freeboard, bank erosion, efficiency reduction and damage of equipment in the plant. On the other hand, previous studies have revealed that undesirable waves produced in either end of the channel can be effectively canceled by a suitable flow control in the opposite end [9]. The problem can be regarded as an optimization problem to determine the optimum control which should be applied at one end to minimize the water surface fluctuations in the channel. Atanov et al. [10] proposed a variational approach for minimizing water-level deviations from a desired value to find an optimum inflow hydrograph (upstream control). They idealized the problem for a frictionless channel with trapezoidal cross-section and followed a fairly complicated procedure using calculus of variations techniques to find a solution for such simplified problem. Unfortunately, it is impossible to extend their procedure for more general cases and an alternate simpler way should be sought.

Powerful metaheuristic algorithms such as ant colony and particle swarm optimizations have been already used in solving difficult engineering problems successfully. Recently, a metaheuristic firefly algorithm has been proposed by Yang [11]. Implementation of this algorithm in water engineering problems has not yet been tested. Firefly algorithm has some advantages such as simplicity and intrinsic capability of finding local optimums. Most water engineering optimization problems can be solved by metaheuristic algorithms, if the problem can be appropriately transformed to an optimization problem suited to these kinds of algorithms.

In this paper, the problem of finding optimal upstream control of a channel containing two pumping stations in both ends is solved by an innovative metaheuristic approach. The problem is first defined as an optimization problem using Fourier series. The unknown coefficients of the series can be next determined by firefly algorithm. The procedure simultaneously employs a shock-capturing unsteady flow solver based on TVD-MacCormack scheme and firefly optimizer module. Both modules have been programmed in digital visual FORTRAN environment whereas MATLAB software has been employed for visualization of the results. The results show satisfactory improvement in the value of the objective function found by Atanov et al. [10]. Moreover, since the flow solver module of the proposed method is general, the previous limitations assumed for flow and channel conditions are practically removed in current study.

2. PROBLEM FORMULATION

The channel shown in Figure 1 has a finite length of *L*. The non-uniform water depth for steady state flow is $H_0(x)$ in which x denotes the longitudinal direction. In unsteady flow case, the channel depth (and velocity) is a function of both distance x and time t and it can be denoted by $H(x, t)$. The unsteady flow is governed by well-known Saint Venant equations as follows [12]:

$$\frac{\partial Q}{\partial t}+\frac{\partial A}{\partial x}=0 \tag{1}$$

$$\frac{\partial H}{\partial t}+\frac{\partial}{\partial x}(QV+gA\overline{H})=gA(S_0-S_f) \tag{2}$$

in which $A(x, t)$ is the cross sectional area of flow, $Q(x, t)$ is discharge, $V(x, t)$ is flow velocity, $\bar{y}$ is the vertical distance between water surface and the centroid of the cross section, S_0 is the channel's longitudinal slope, S_f is the energy slope that represents the effect of friction and g is gravitational acceleration. The first equation is continuity equation and the second one is momentum equation.

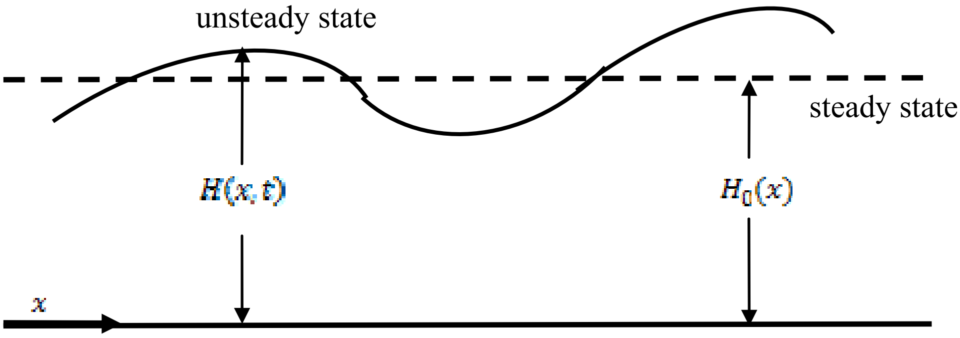

Figure 1. Definition of parameters in open channel flow

Either water depth or discharge must be specified on the ends of the domain to determine the water surface profile within the domain. For unsteady flow, the problem is subjected to following initial conditions:

$$Q(x,0)=\phi_1(x)$$
$$H(x,0)=\phi_2(x) \tag{3a, b}$$

where both $\phi_1(x)$ and $\phi_2(x)$ are known functions since initial conditions are specified. The channel is bounded by pumping stations on both ends, imposing boundary conditions on discharge as:

$$Q(0,t)=\eta(t)$$
$$Q(L,t)=\psi(t) \tag{4a, b}$$

It is assumed that the operation of downstream pumping station is specified and hence the function $\psi(t)$ is known. For specified function $\psi(t)$ acting on downstream end, the problem is to find the optimal control (or controlling function) $\eta(t)$ subject to governing equations (1) and (2), initial and boundary conditions (3) , (4b), such that the water level deviation from the initial and desired water depth $H_0(x)$ is minimized.

3. DEFINING THE PROBLEM AS A METAHEURISTIC OPTIMIZATION PROBLEM

The problem defined in the previous section can be solved via different approaches. Atanov et al. [10] developed a variational approach to solve the problem with some simplifications such as assuming frictionless channel with trapezoidal cross section. They followed a relatively complicated procedure to find an optimum control for $\eta(t)$. In this study, by defining the problem as a metaheuristic optimization problem, a more general approach is developed for finding controlling function $\eta(t)$. To achieve this, the first step is to define an objective function for the problem. Mathematically, the objective function is an averaged value of deviations from $H_0(x)$ in all times and distances as:

$$f = \frac{1}{LT}\int_0^T\int_0^L \left|H(x,t) - H_0(x)\right| dx\,dt \tag{5}$$

in which *T* is a given transient duration or duration time over which the solution is sought. Clearly, since $H(x,t)$ depends on boundary condition $\eta(t)$, the value of objective function *f* is also a function of $\eta(t)$.

In a numerical solution of equations (1) and (2), since the domain of the problem is discretized both spatially and temporally, the objective function can be written as follow:

$$f = \sum_{j=1}^{k}\sum_{i=1}^{n}\left|H(x_i,t_j) - H_0(x)\right| \tag{6}$$

where $x_k = L$ and $t_n = T$.

The second step is to approximate controlling function $\eta(t)$ such that it can be estimated and optimized by an optimization algorithm. For duration time *T*, the function $\eta(t)$ can be expanded using Fourier series:

$$\eta(t) = b_0 + \sum_{m=1}^{\infty}\left[b_m\,Cos\left(\frac{m\pi t}{T}\right) + a_m\,Sin\left(\frac{m\pi t}{T}\right)\right] \tag{7}$$

By means of equation (7), one can choose a maximum value for *m* (i.e. the maximum number of sine and cosine terms in the Fourier series) and optimize these values by an optimization method. The more terms in Fourier series, the more accurate estimation of $\eta(t)$. However, it is worth pointing out that choosing a large number of optimization parameters in any optimization algorithm expands the searching space extensively and consequently the algorithm may be unable to find a solution.

Defining a vector of optimization variables as unknown coefficients of Fourier series as *X*:

$$X = \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ a_m \\ b_0 \\ b_1 \\ . \\ . \\ b_m \end{bmatrix} \tag{8}$$

the task is to find the best vector *X* that minimizes the objective function in Eq. (6).

After some trials, *m*=5 was adopted in this research. Therefore, the optimization vector *X* contains total number of 11 unknown coefficients $a_1, a_2, a_3, a_4, a_5, b_0, b_1, b_2, b_3, b_4, b_5$ that should be optimized.

4. AN OVERVIEW ON FIREFLY OPTIMIZATION ALGORITHM

The Firefly Algorithm (FA) is one of the latest metaheuristic algorithms. Firefly algorithm is a nature-inspired algorithm, which was first developed by Yang [11] inspired by the light attenuation over the distance and fireflies' mutual attraction. Algorithm considers what each firefly observes at the point of its position, when trying to move to a greater light-source, than is his own. Firefly algorithm idealizes some of the characteristics of the firefly behavior in nature. They follow three rules: i) all the fireflies are unisex, ii) attractiveness is proportional to their flashing brightness which decreases as the distance from the other firefly increases due to the fact that the air absorbs light. Since the most attractive firefly is the brightest one, to which it convinces neighbors moving toward. In case of no brighter one, it freely moves any direction and, iii) brightness of every firefly determines its quality of solution; in most of the cases, it is proportional to the objective function.

The main steps of the FA start from initializing a swarm of fireflies, each of which is determined the flashing light intensity. During the loop of pairwise comparison of light intensity, the firefly with lower light intensity will move toward the higher one. The moving distance depends on the attractiveness. After moving, the new firefly is evaluated and updated for the light intensity. During pairwise comparison loop the best-so-far solution is iteratively updated. The pairwise comparison process is repeated until termination criteria are satisfied. Finally, the best-so-far solution is visualized.

To define the most important parameters in firefly algorithm suppose it is a night with absolute darkness, where the only visible light is the light produced by fireflies. The light intensity of each firefly is proportional to the quality of the solution, it is currently located at. In order to improve own solution, the firefly needs to advance towards the fireflies that have brighter light emission than is his own. Each firefly observes decreased light intensity, than the one firefly actually emit, due to the air absorption over the distance.

Attractiveness of a firefly abides the law [11]:

$$\beta = \beta_0 \exp(-\gamma r) \tag{9}$$

in which β_0 is the attractiveness in distance *r=0* and γ is light absorption coefficient in the range $[0,\infty)$. The distance *r* between firefly *i* and *j* at x_i and x_j is defined as Cartesian distance:

$$r = r_{ij} = \left\| x_i - x_j \right\| = \sqrt{\sum_{k=1}^{d} (x_{i,k} - x_{j,k})^2} \tag{10}$$

where $x_{i,k}$ is the k^{th} component of the spatial coordinate x_i of the i^{th} firefly and *d* is the number of dimensions. Moreover, the movement of firefly *i* which is attracted by a more attractive or brighter firefly *j* is given by the following equation:

$$x_i = x_i + \beta_0 \exp(-\gamma r^2)(x_j - x_i) + \alpha(\varepsilon - 0.5) \tag{11}$$

where the second term is due to the attraction. The third term is randomization with α being the randomization parameter such that α ∈ [0, 1], and ε is a vector of random numbers drawn from a Gaussian distribution or uniform distribution in the range [0, 1]. Furthermore, for most problems, one can take $\beta_0 = 1$

5. UNSTEADY FLOW SIMULATION

In order to find the optimal control $\eta(t)$ which minimizes the water surface fluctuations *f* in Eq. (6) for specified time interval *T* and channel length *L*, an unsteady non-uniform flow simulation with shock-capturing ability is necessary. The shock-capturing ability requirement of the numerical scheme is because of development of strong shocks in the flow domain due to abrupt changes in flow rate at upstream and downstream pumping stations. On the other hand, since the simulation of flow should be accomplished several times in the optimization procedure, a robust and satisfactory accurate model with low computational cost is recommended. In this study a TVD-MacCormack scheme is adopted for this purpose. This is a shock-capturing scheme with low computational cost compared to other numerical schemes and suitable accuracy.

To simulate flow with TVD-MacCormack scheme, first, the one-dimensional governing equations (1) and (2) are written as:

$$\frac{\partial S}{\partial t}+\frac{\partial F}{\partial x}=C \tag{12}$$

in which

$$S=\begin{pmatrix}A\\VA\end{pmatrix},\quad F=\begin{pmatrix}VA\\V^2A+gA\bar{y}\end{pmatrix},\quad C=\begin{pmatrix}0\\gA(S_0-S_f)\end{pmatrix} \tag{13}$$

The vector S is the vector of conserved variables, F is flux vector and C is the source term. The energy slope can be expressed by means of Manning roughness coefficient n as:

$$S_f=\frac{n^2V^2}{R^{4/3}} \tag{14}$$

TVD-MacCormack scheme combines the well known MacCormack scheme with total variation diminishing (TVD) approach to build a robust high-resolution scheme. The MacCormack scheme is a two-step predictor-corrector scheme. In the predictor stage, a backward difference discretization is used whereas in the corrector stage a forward difference discretization is employed. This type of discretization well agrees with both upstream and downstream travelling waves associated with positive and negative eigenvalues of the problem. A TVD stage of the procedure has been added to the MacCormack scheme by Davis [13] and has been developed by Mingham et al. [14] and Liang et al. [15]. The whole procedure can be summarized as follows:

The predictor step:

$$S_i^p=S_i^n-(F_i^n-F_{i-1}^n).\Delta t/\Delta x+S^n.\Delta t \tag{15a}$$

The corrector step:

$$S_i^c=S_i^n-(F_{i+1}^p-F_i^p).\Delta t/\Delta x+S^p.\Delta t \tag{15b}$$

The TVD step:

$$S_i^{n+1}=0.5(S_i^p+S_i^c)+[G(r_i^+)+G(r_{i+1}^-)].\Delta S_{i+1/2}^n-[G(r_{i-1}^+)+G(r_i^-)].\Delta S_{i-1/2}^n \tag{15c}$$

in which $\boldsymbol{\Delta x}$ and $\boldsymbol{\Delta t}$ are the spatial and temporal steps, respectively, the superscript $\boldsymbol{n}$ denotes time step, the subscript $\boldsymbol{i}$ denotes the node number. Moreover, the following relations hold:

$$\Delta S_{i+1/2}^n=S_{i+1}^n-S_i^n,\ \Delta S_{i-1/2}^n=S_i^n-S_{i-1}^n \tag{16a, b}$$

$$r_i^+=\frac{<\Delta S_{i-1/2}^n,\Delta S_{i+1/2}^n>}{<\Delta S_{i+1/2}^n,\Delta S_{i+1/2}^n>},\quad r_i^-=\frac{<\Delta S_{i-1/2}^n,\Delta S_{i+1/2}^n>}{<\Delta S_{i-1/2}^n,\Delta S_{i-1/2}^n>} \tag{17a, b}$$

The brackets indicate the scalar product of the two vectors in the bracket.

The function G which has been employed to ensure TVD property of the scheme is defined as:

$$G(x) = 0.5 \times \sigma \times [1 - \phi(x)] \tag{18}$$

in which the flux limiter function $\varphi(x)$ has been employed to suppress the spurious numerical oscillations and is defined as follows:

$$\varphi(x) = max(0, min(2x, 1)) \tag{19}$$

and the parameter σ is:

$$\sigma = \begin{cases} CFL \times (1 - CFL) & CFL \le 0.5 \\ 0.25 & CFL > 0.5 \end{cases} \tag{20}$$

in which *CFL* is the well known Courant-Friedrichs-Lewy number defined as:

$$CFL = \frac{\Delta t}{\Delta x}(|u| + \sqrt{gh}) \tag{21}$$

The overall proposed procedure to solve the problem of optimization of water surface fluctuations by firefly algorithm has been shown in Table 1 as a pseudo code.

Table 1. The pseudo code to find optimal controlling function using firefly algorithm

```
Set k = 1;
Randomly initialize the solution vector x_i^(k).
Light intensity f_i at x_i^(k) is determined by f( x_i^(k) ) using Unsteady Flow Simulator
Define light absorption coefficient γ
WHILE (the termination conditions are not met)
  define η(t) for each vector x_i^(k) using Eq.(7);
      FOR i=1:n   (all n fireflies, i.e. all n candidate curves)
        Call Unsteady Flow Simulator to obtain f_i
     END FOR

      FOR i=1:n   (all n fireflies)
          FOR j=1:n   (all n fireflies)
               Compute distance r between firefly i and j
               If ( f_i > f_j ),  Move firefly i towards j using Eq.(11); end if
               Vary attractiveness with distance r
          END FOR j
      END FOR i
Rank the fireflies and find the current global best
Set k = k + 1
END WHILE
```

6. DESIGN EXAMPLE

The problem solved by Atanov et al. [10] is considered here as an example to implement the proposed technique and compare the results.

To define the problem, consider steady state flow in a channel with rectangular cross section and 20km long and 30m wide. The flow rates of the supplying (upstream) and withdrawing (downstream) pumping stations are initially the same. The initial and desired water depth in the channel is 3.6 m with initial flow rate equal to 100 m^3/s. The unsteady flow in the problem starts by increase in the downstream pumping station withdrawal rate. The downstream flow demand is increased by 50% so that $\psi = 150\, m^3 / s$. The flow was simulated by the method described in section 5 with $\Delta x = 400\, m$. The required time steps were automatically calculated by means of Eq. (21) and the total simulation time was 4 hours.

If the controlling upstream flow rate, $\eta(t)$ is not optimized, the only way to maintain the water level in the channel is obviously matching the downstream flow rate, i.e., to also increase the flow rate at the upstream (supplying) pumping station by 50%. Another strategy is to impose a control on upstream pumping station so that the introduced waves cancel the waves that travel in opposite directions along the channel. Atanov et al. [10] obtained an optimal curve by variational approach to do this. Figure 2 shows the upstream flow rates for the two cases (with control shown with solid line, and without control shown in dashed line). When no control is imposed at upstream end, the water surface elevation time series at the two ends of the canal are as shown in Figure 3. As shown in the Figure 3, abrupt change of the flow rates at both pumping stations at two ends develops waves which move in opposite directions along the channel. A positive wave moves downstream, while a negative wave moves upstream. However, the waves do not simply cancel each other. When the waves reach the end of the channel, they reflect. This process develops remarkable fluctuations within the channel as shown in 3D view in Figure 4. The value of objective function in this case is the considerable value of 2396.33 m for full length of channel and over the simulation time. The results of imposing the control found by Atanov et al. [10] are included for the sake of comparison. If this upstream control (shown by solid line in Figure 2) is imposed, the desired flow rate starts at a larger value than the downstream flow rate as shown in the Figure. This sounds to be necessary to compensate for the negative wave moving upstream. The upstream flow rate then oscillates around the value corresponding to the downstream flow rate. The time series of water level fluctuations at both ends are depicted in Figure 5 for this case. According to the length of the channel, approximately 1 hour is required for two waves to contact each other. After this time, the fluctuations of the water level are small compared to the case of no upstream control (Figure 3) and around to the desired water level as shown in Figure 5. The 3D view of water level fluctuations is given in Figure 6. Obviously, the water-level fluctuations at the pumping stations and whole channel are reduced when the upstream flow rate is optimized using the variational approach. The value of objective function has been considerably reduced to 1107.39 m by imposition of a suitable control and the water level has been stabilized.

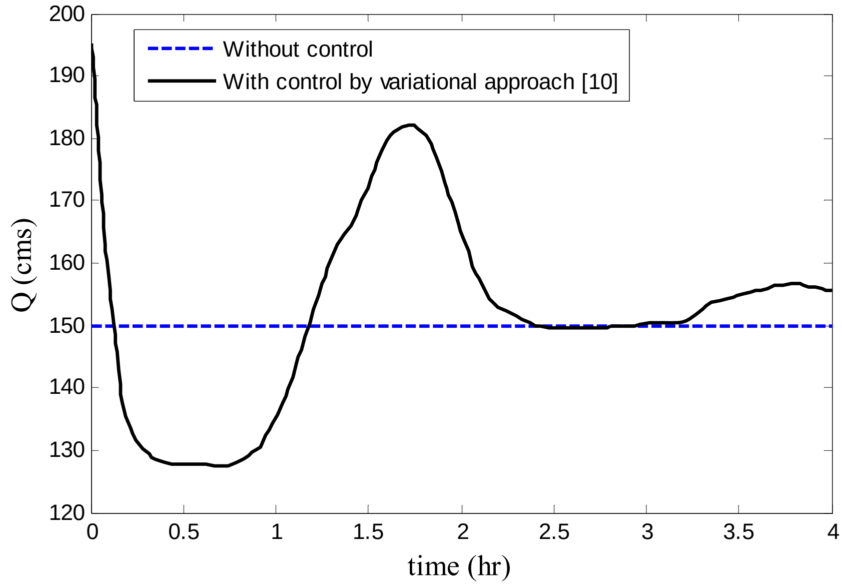

Figure 2. Optimal control found in [10] versus no control at upstream end

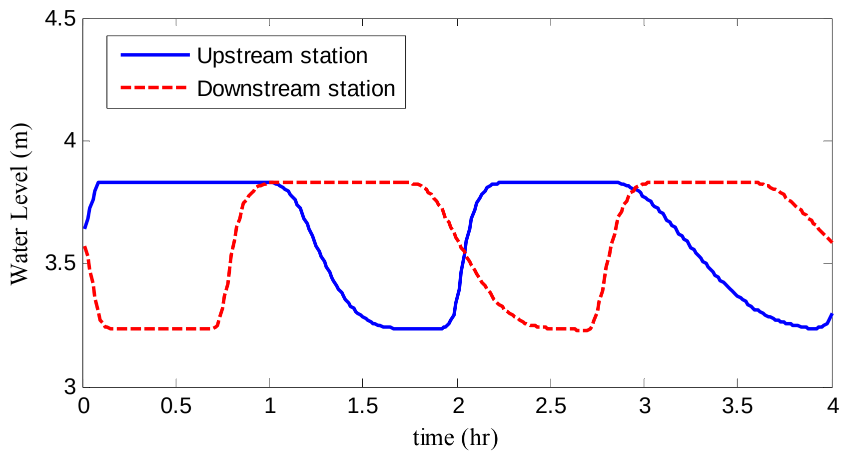

Figure 3. Upstream and downstream time series of water surface level when no control (dashed line in Figure 2) is imposed at upstream end

In the next step, the method described in this study was used to find the optimal control. After various runs, the following design variables were found by the proposed approach with minimum value of objective function equal to 1015.58 m:

$$X = [-3697.99 \ -714.694 \ 1226.99 \ 288.973 \ -71.6161 \ 2253.45 \ 501.998 \\ -2498.24 \ -580.882 \ 415.061 \ 86.1951]^T \tag{22}$$

which offers the following optimal curve:

$$\eta(t) = -3697.99\,Sin\left(\frac{\pi\,t}{T}\right) - 714.694\,Sin\left(\frac{2\pi\,t}{T}\right) + 1226.99\,Sin\left(\frac{3\pi\,t}{T}\right) + 288.973\,Sin\left(\frac{4\pi\,t}{T}\right)$$
$$-71.6161\,Sin\left(\frac{5\pi\,t}{T}\right) + 2253.45 + 501.998\,Cos\left(\frac{\pi\,t}{T}\right) - 2498.24\,Cos\left(\frac{2\pi\,t}{T}\right) \quad (23)$$
$$-580.882\,Cos\left(\frac{3\pi\,t}{T}\right) + 415.061\,Cos\left(\frac{4\pi\,t}{T}\right) + 86.1951\,Cos\left(\frac{5\pi\,t}{T}\right)$$

in which *T=14400 sec* is total simulation time.

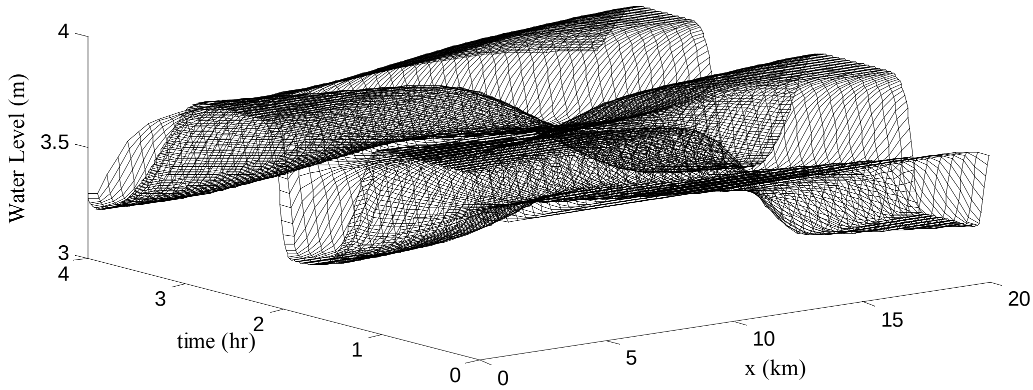

Figure 4. A 3D view of water level fluctuations in the whole channel over simulation time when no control (dashed line in Figure 2) is imposed at upstream end (*f=2396.33 m*)

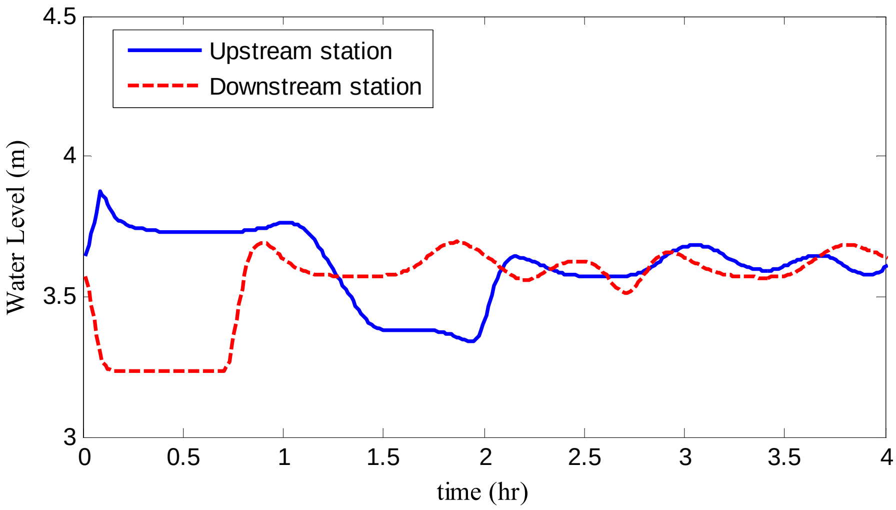

Figure 5. Upstream and downstream time series of water surface level when the control found in [10] (solid line in Figure 2) is imposed at upstream end

The curve defined by Eq. (23) is depicted in Figure 7. The curve found by the proposed method has the similar trend of the curve found by variational approach (Figure 2). However, the value of objective function is less while imposing Eq. (23) at upstream end rather than imposing the curve shown in Figure 2. The optimal curve found by the proposed approach has reduced the water level fluctuations by 91.81 m (approximately 9% smaller than variational approach). The effect of imposing the optimal curve of Eq. (23) at upstream end on time series of water surface fluctuations at both ends is shown in Figure 8. As it is clear in this figure, the fluctuations have been reduced compared to Figure 3 and Figure 5. The water level has been more stabilized after 2 hours compared to Figure 5 which indicates the better performance of the proposed optimal curve. For further evidence, 3D view of water surface fluctuations has been shown in Figure 9. Compared to imposition of optimal curve found by variational approach, the proposed optimal control shows fewer fluctuations.

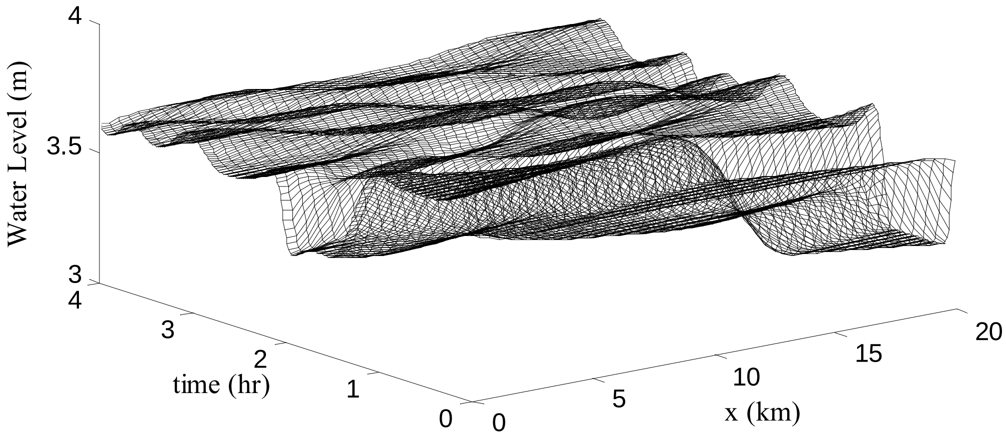

Figure 6. A 3D view of water level fluctuations in the whole channel over simulation time when the control found in [10] (solid line in Figure 2) is imposed at upstream end (*f=1107.39 m*)

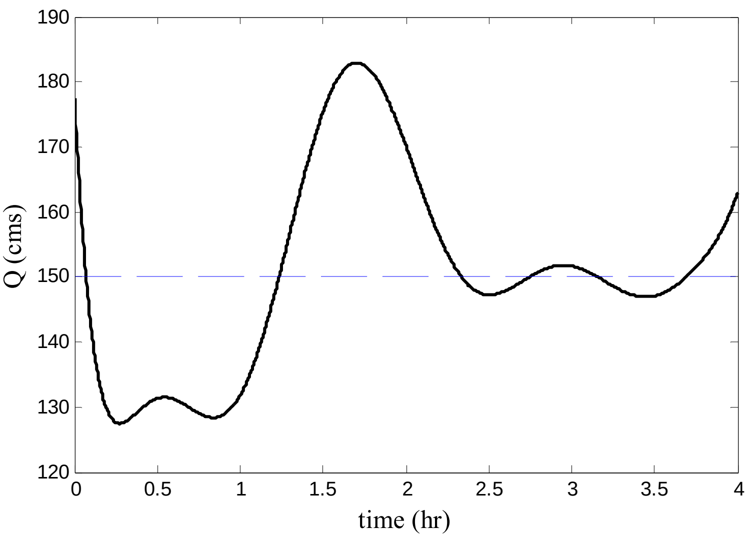

Figure 7. Upstream control found by proposed approach (solid line)

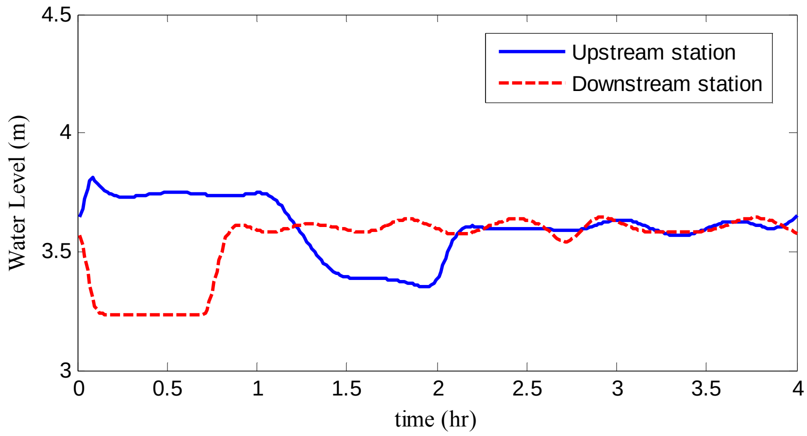

Figure 8. Upstream and downstream time series of water surface level when the control found in this study (solid line in Figure 7) is imposed at upstream end

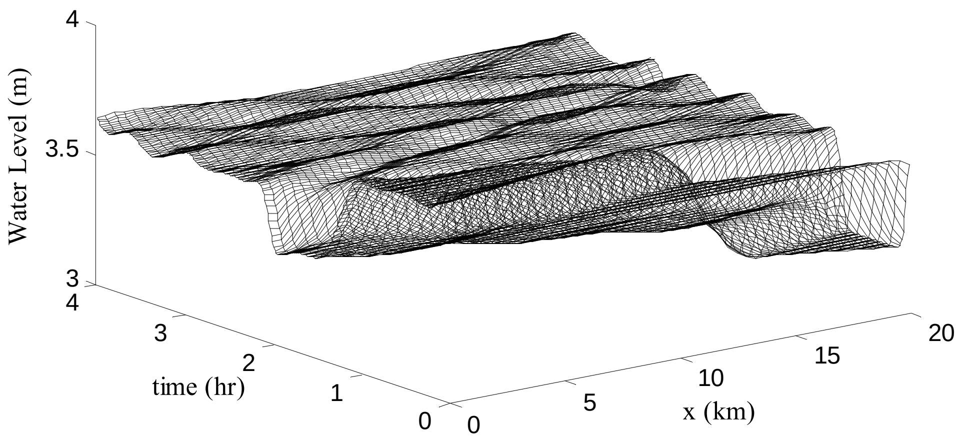

Figure 9. A 3D view of water level fluctuations in the whole channel over simulation time when the control found in this study (solid line in Figure 7) is imposed at upstream end ($f=1015.58$ *m*)

7. SUMMARY AND CONCLUSION

The optimal control on pumping stations in conveyance open channels was found by a metaheuristic firefly algorithm. The process involves combining a shock-capturing flow simulator, Fourier series and firefly optimization algorithm are utilized to obtain the optimum values of the coefficients of the Fourier series expansion of optimal curve. The results showed that the proposed method is very effective and much simpler than the direct mathematical methods such as variational approach. Better results may be found by considering more terms in the Fourier series. However, the size of the searching space will increase and it will need more efforts to find the optimal values. The idea of expanding the desired curve by Fourier series and finding the optimum coefficients may also be applied to similar problems.

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